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A general construction of fractal interpolation functions on grids of $\mathbb{R}^n$

P. BOUBOULIS[1] and L. DALLA[2]

[1]*Department of Informatics and Telecommunications, University of Athens, Panepistimiopolis 157 84, Athens, Greece*
email: bouboulis@di.uoa.gr
[2]*Department of Mathematics, University of Athens, Panepistimiopolis 157 84, Athens, Greece*
email: ldalla@math.uoa.gr

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We generalise the notion of fractal interpolation functions (FIFs) to allow data sets of the form

$$\{(x_{1,i_1}, x_{2,i_2}, \ldots, x_{n,i_n}, z_{i_1,i_2,\ldots,i_n}); i_k = 0, 1, \ldots, N_k,\ k = 1, 2, \ldots, n\} \subset I \times \mathbb{R},$$

where $I = [0, 1]^n$. We introduce recurrent iterated function systems whose attractors G are graphs of continuous functions $f : I \to \mathbb{R}$, which interpolate the data. We show that the proposed constructions generalise the previously existed ones on $\mathbb{R}$. We also present some relations between FIFs and the Laplace partial differential equation with Dirichlet boundary conditions. Finally, the fractal dimensions of a class of FIFs are derived and some methods for the construction of functions of class C^p using recurrent iterated function systems are presented.

1 Introduction

Fractal interpolation is an alternative to traditional interpolation techniques, which gives a broader set of interpolants. Using this method, we can construct not only interpolants with non-integral dimension (as its name implies) but also smooth, non-polynomial interpolants, or even splines and Hermite functions (however, most applications make use of the fractal construction). Its main differences with the traditional interpolation techniques consist (a) in the definition of a functional relation (see equation (3.5)), that implies a self similarity in small scales; (b) in the constructive way (through iterations), that it is used to compute the interpolant, instead of the descriptive one (usually a formula) provided by the classical methods and (c) in the usage of some parameters, which we call vertical scaling factors, that define the dimension of the interpolant. Fractal interpolation functions (FIFs) are highly irregular and cannot be described using elementary functions such as polynomials (excluding the trivial cases where the fractal function is actually a spline or some other ordinary interpolant). As we mentioned above, they are constructed through iterations, starting with an arbitrary function. The construction uses the well-known, in the fractal literature, Deterministic Iterative Algorithm. This algorithm is often used to construct fractal sets, which are determined through an iterated function system (IFS)

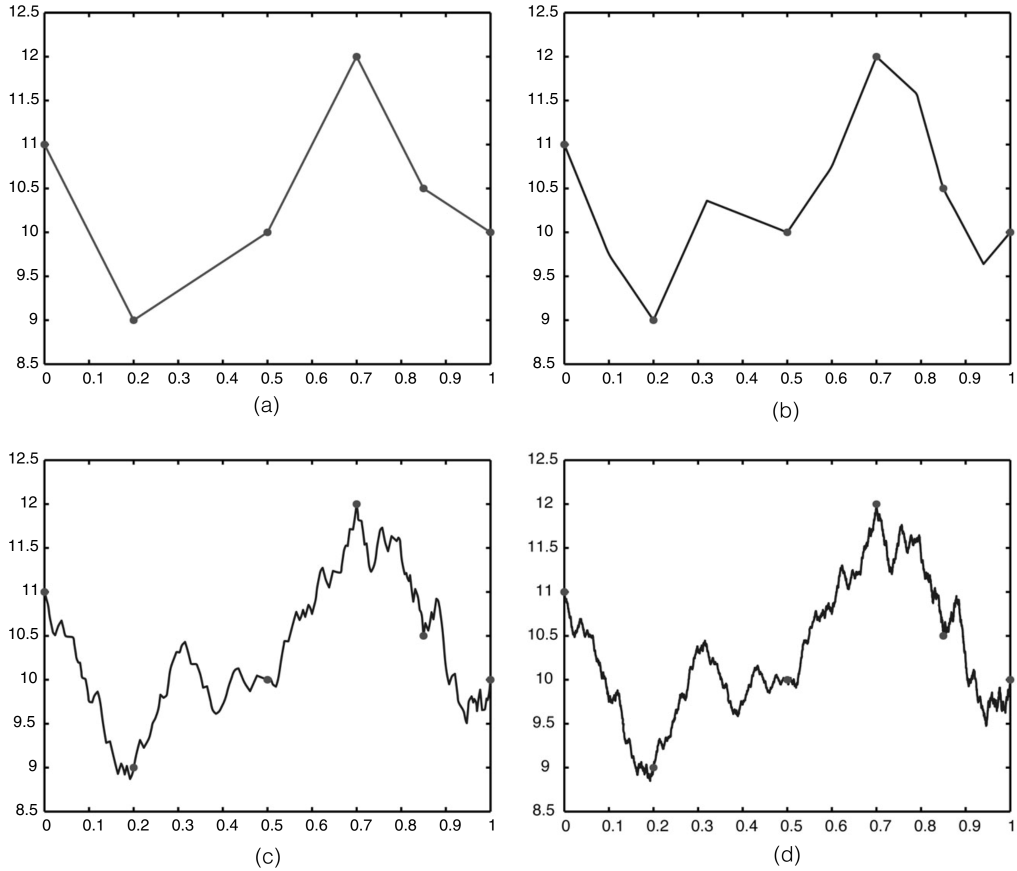

FIGURE 1. The construction of a fractal interpolation function: (a) the initial arbitrary function; (b) after one iteration; (c) after five iterations; and (d) after 15 iterations, the approximation of the fractal interpolant is sufficient.

(a pair consisting of a complete metric space (X, ρ) together with a finite set of continuous, contractive mappings – see Section 2 for more details). The algorithm starts with an arbitrary, compact set and applies the maps of the IFS to the set successively (Figure 1). To obtain the actual fractal interpolant, one need to continue the iterations indefinitely. However, a small number of iterations usually gives a sufficient approximation.

Fractal interpolation functions have been used in approximation theory, in the modelling of one-dimensional signals (especially in the case of signals that are highly irregular), in computer graphics (to construct coastlines, mountain lines, surfaces), in the modelling of the surfaces of minerals, in image compression, in remote sensing and in other scientific applications. They were introduced by M. Barnsley [1] as attractors of IFSs. Later in [3], the construction was generalised with the help of recurrent iterated function systems (RIFSs). Both constructions, however, involve data points of the form $\{(x_i, z_i) \in I \times \mathbb{R};\ i = 0, 1, \ldots, N\}$, where $I = [0, 1]$. In the years that followed, several problems of the construction of FIF were addressed by Barnsley and others. For example, the box-counting dimension of some types of FIFs were studied in [1, 3, 4, 9, 14], the integrability and differentiability were studied in [5] and some other interesting issues were raised in [10, 13, 15, 18]. Recently, it has been demonstrated that FIFs may be used to generalise spline functions [7, 19, 20].

Nevertheless, the construction of FIFs remained restricted to the case of one-dimensional data points. Several attempts were made to construct FISs by Massopust and others [6, 12, 16, 17] in the case where the interpolation points or the vertical scaling factors are confined. However, the general problem remains open. In most cases, the construction uses, either interpolation points, which are restricted to be colinear in the borders of $I = [0, 1]^2$, or maps with equal vertical scaling factors. Our intention is to generalise these results to allow arbitrary data points that lie not only on $\mathbb{R}^2$ but also on $\mathbb{R}^n$. Thus, in this article, we introduce a way of constructing FIFs that interpolate given data points of the form $\{(x_{1,i_1}, x_{2,i_2}, \ldots, x_{n,i_n}, z_{i_1,i_2,\ldots,i_n}) \in I \times \mathbb{R}; i_k = 0, 1, \ldots, N_k,\ k = 1, 2, \ldots, n\}$, where $I = [0, 1]^n$. The proposed construction makes use of ordinary interpolants of points of $\mathbb{R}^{n-1}$ to generate the fractal interpolant of some predefined points of $\mathbb{R}^n$. Section 2 contains the mathematical background on RIFS and Section 3 the main theorems, which we deduced for the construction of FIFs. In Section 4, we present some special cases using the theorems of Section 3. Next, in Section 5, the box-counting dimension for a class of FIFs is determined to ensure that the proposed construction gives indeed fractal surfaces. Finally, in Section 6, we present some theorems that enable us to construct C^p FIFs.

2 Recurrent iterated function systems

A *hyperbolic IFS* is defined as a pair consisting of a complete metric space (X, ρ) together with a finite set of continuous contractive mappings $w_i : X \to X$, with respective contraction factors s_i for $i = 1, 2, \ldots, N$ $(N \geqslant 2)$. The attractor of a hyperbolic IFS is the unique set E for which $E = \lim_{k\to\infty} W^k(A_0)$ for every starting compact set A_0, where

$$W(A) = \bigcup_{i=1}^{N} w_i(A) \text{ for all } A \in \mathscr{H}(X),$$

and $\mathscr{H}(X)$ is the complete metric space of all non-empty compact subsets of X with respect to the Hausdorff metric h. Iterated function systems are able to produce very complicated attractors using only a handful of mappings.

A more general concept, which allows the construction of even more complicated sets, is that of the *RIFS*, which consists of the IFS $\{X; w_i,\ i = 1, 2, \ldots, N\}$ (or more briefly $\{X; w_{1\text{—}N}\}$), together with an irreducible row-stochastic matrix $P\ :=(p_{v,\mu} \in [0, 1]\ :\ v, \mu = 1, \ldots, N)$, such that

$$\sum_{\mu=1}^{N} p_{v,\mu} = 1, \quad v = 1, \ldots, N. \tag{2.1}$$

The recurrent structure is given by the (irreducible) *connection matrix* $C := (C_{v,\mu}\ :\ v, \mu = 1, 2, \ldots, N)$, which is defined by

$$C_{v,\mu} = \begin{cases} 1, & \text{if } p_{\mu,v} > 0 \\ 0, & \text{if } p_{\mu,v} = 0 \end{cases},$$

where $\nu, \mu = 1, 2, \ldots, N$. The transition probability for a certain discrete time Markov process is $p_{\nu,\mu}$, which gives the probability of transfer into state μ given that the process is in state ν. Condition (2.1) says that whichever state the system is in (say ν), a set of probabilities is available that sum to 1 and describe the possible states to which the system transits at the next step.

We define the mappings

$$W_{i,j} : \mathscr{H}(X) \to \mathscr{H}(X), \text{ with } W_{i,j}(A) = \begin{cases} w_i(A), & p_{j,i} > 0 \\ \emptyset, & p_{j,i} = 0 \end{cases}, \tag{2.2}$$

for all $A \in \mathscr{H}(X)$ and the metric space

$$\tilde{\mathscr{H}}(X) = \mathscr{H}(X)^N = \mathscr{H}(X) \times \mathscr{H}(X) \times \cdots \times \mathscr{H}(X) \tag{2.3}$$

equipped with the metric

$$\tilde{h}\left(\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix}, \begin{pmatrix} B_1 \\ B2 \\ \vdots \\ B_N \end{pmatrix}\right) = \max\{h(A_i, B_i); i = 1, 2, \ldots, N\}.$$

Next, we define the map

$$\boldsymbol{W} : \tilde{\mathscr{H}}(X) \to \tilde{\mathscr{H}}(X) : \boldsymbol{W}\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \ldots & W_{1N} \\ W_{21} & W_{22} & \ldots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{N1} & W_{N2} & \ldots & W_{NN} \end{pmatrix} \cdot \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix} = \begin{pmatrix} \bigcup_{j \in I(1)} w_1(A_j) \\ \bigcup_{j \in I(2)} w_2(A_j) \\ \vdots \\ \bigcup_{j \in I(N)} w_N(A_j) \end{pmatrix},$$

where $I(i) = \{j : p_{j,i} > 0\}$, for $i = 1, 2, \ldots, N$. If w_i are contractions, then $\boldsymbol{W}$ is a contraction and there is $\boldsymbol{E} = (E_1, E_2, \ldots, E_N)^t \in \tilde{\mathscr{H}}(X)$ such that $\boldsymbol{W}(\boldsymbol{E}) = \boldsymbol{E}$ and $E_i = \bigcup_{j \in I(i)} w_i(E_j)$, for $i = 1, 2, \ldots, N$. Then, the set $G = \bigcup_{i=1}^{N} E_i$ is called the attractor of the RIFS $\{X,\ w_{1\text{—}N},\ P\}$. Evidently,

$$G = \lim_n A_n.$$

Let $A \in \mathscr{H}(X)$. We define the sequences $\{\boldsymbol{A}_n\}_{n \in \mathbb{N}}$ in $\tilde{\mathscr{H}}(X)$ and $\{A_n\}_{n \in \mathbb{N}}$ in $\mathscr{H}(X)$ as follows: $\boldsymbol{A}_0 = (A, A, \ldots, A)^t$, $\boldsymbol{A}_n = \boldsymbol{W}(\boldsymbol{A}_{n-1})$ and $A_n = \bigcup_{i=1}^{N} (\boldsymbol{A}_n)_i$, for $n \in \mathbb{N}$, where $\boldsymbol{A}_n = ((\boldsymbol{A}_n)_1, (\boldsymbol{A}_n)_2, \ldots, (\boldsymbol{A}_n)_N)$.

3 Fractal interpolation functions

Consider a data set

$$\Delta = \{(x_{1,i_1}, x_{2,i_2}, \ldots, x_{n,i_n}, z_{i_1,i_2,\ldots,i_n}) \in I \times \mathbb{R}; i_k = 0, 1, \ldots, N_k,\ k = 1, 2, \ldots, n\},$$

such that $0 = x_{k,0} < x_{k,1} < \cdots < x_{k,N_k} = 1$, $N_k \in \mathbb{N}$, for $k = 1, 2, \ldots, n$, where $I = [0,1]^n$, which contains in total $(N_1 + 1) \cdot (N_2 + 1) \cdots (N_n + 1) = \prod_{k=1}^{n} (N_k + 1)$ points. In addition, consider a data set $\hat{\Delta} \subset \Delta$

$$\hat{\Delta} = \{(\hat{x}_{1,j_1}, \hat{x}_{2,j_2}, \dots, \hat{x}_{n,j_n}, \hat{z}_{j_1,j_2,\dots,j_n}) \in I \times \mathbb{R}; j_k = 0,1,\dots,M_k,\ k = 1,2,\dots,n\},$$

such that $0 = \hat{x}_{k,0} < \hat{x}_{k,1} < \dots < \hat{x}_{k,M_k} = 1$, $M_k \in \mathbb{N}$ for $k = 1,2,\dots,n$, which contains in total $\prod_{k=1}^{n}(M_k + 1)$ points. To simplify the notation, we set

$$\boldsymbol{i} = (i_1, i_2, \dots, i_n) \in \{0,1,\dots,N_1\} \times \{0,1,\dots,N_2\} \times \cdots \times \{0,1,\dots,N_n\} = \mathbb{A}_0,$$
$$\boldsymbol{j} = (j_1, j_2, \dots, j_n) \in \{0,1,\dots,M_1\} \times \{0,1,\dots,M_2\} \times \cdots \times \{0,1,\dots,M_n\} = \mathbb{B}_0,$$

and

$$\mathbb{A}_1 = \{1,2,\dots,N_1\} \times \{1,2,\dots,N_2\} \times \cdots \times \{1,2,\dots,N_n\},$$
$$\mathbb{B}_1 = \{1,2,\dots,M_1\} \times \{1,2,\dots,M_2\} \times \cdots \times \{1,2,\dots,M_n\}.$$

Thus, we may rewrite Δ and $\hat{\Delta}$ as follows:

$$\Delta = \{(\boldsymbol{x_i}, z_{\boldsymbol{i}}) \in I \times \mathbb{R},\ \boldsymbol{i} \in \mathbb{A}_0\}, \qquad \text{where } \boldsymbol{x_i} = (x_{1,i_1}, x_{2,i_2}, \dots, x_{n,i_n}) \in I,$$
$$\hat{\Delta} = \{(\hat{\boldsymbol{x}}_{\boldsymbol{j}}, \hat{z}_{\boldsymbol{j}}) \in I \times \mathbb{R},\ \boldsymbol{j} \in \mathbb{B}_0\}, \qquad \text{where } \hat{\boldsymbol{x}}_{\boldsymbol{j}} = (\hat{x}_{1,j_1}, \hat{x}_{2,j_2}, \dots, \hat{x}_{n,j_n}) \in I.$$

We also define the sets

$$\Delta' = \{\boldsymbol{x_i};\ \boldsymbol{i} \in \mathbb{A}_0\}, \quad \hat{\Delta}' = \{\hat{\boldsymbol{x}}_{\boldsymbol{j}};\ \boldsymbol{j} \in \mathbb{B}_0\}.$$

Let $\langle \boldsymbol{e}_{n,k}; k = 1,2,\dots,n\rangle$ be the standard basis of $\mathbb{R}^n$. Furthermore, for any $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$, we use the notations $\mathrm{proj}_{-\lambda}\boldsymbol{x}$, $\mathrm{proj}_{\lambda}\boldsymbol{x}$ as follows:

$$\mathrm{proj}_{-\lambda}\boldsymbol{x} = (x_1, x_2, \dots, x_{\lambda-1}, x_{\lambda+1}, \dots, x_n) \in \mathbb{R}^{n-1},$$
$$\mathrm{proj}_{\lambda}\boldsymbol{x} = (x_1, x_2, \dots, x_{\lambda-1}, 0, x_{\lambda+1}, \dots, x_n) \in \mathbb{R}^n.$$

The points of Δ' divide $[0,1]^n$ into $\prod_{k=1}^{n} N_k$ regions

$$I_{\boldsymbol{i}} = [x_{1,i_1-1}, x_{1,i_1}] \times [x_{2,i_2-1}, x_{2,i_2}] \times \cdots \times [x_{n,i_n-1}, x_{n,i_n}],$$

for all $\boldsymbol{i} = (i_1, i_2, \dots, i_n) \in \mathbb{A}_1$, while the points of $\hat{\Delta}'$ divide $[0,1]^n$ into $\prod_{k=1}^{n} M_k$ domains

$$J_{\boldsymbol{j}} = [\hat{x}_{1,j_1-1}, \hat{x}_{1,j_1}] \times [\hat{x}_{2,j_2-1}, \hat{x}_{2,j_2}] \times \cdots \times [\hat{x}_{n,j_n-1}, \hat{x}_{n,j_n}],$$

for all $\boldsymbol{j} = (j_1, j_2, \dots, j_n) \in \mathbb{B}_1$. We make the additional assumption that for every $\boldsymbol{j} \in \mathbb{B}_1$, there is at least one interpolation point that lies in the interior of $J_{\boldsymbol{j}} \times \mathbb{R}$. Furthermore, we define a labelling map $\mathscr{J} : \mathbb{A}_1 \to \mathbb{B}_1 : \mathscr{J}(\boldsymbol{i}) = \boldsymbol{j}$ and the 1–1 functions Φ and $\hat{\Phi}$ (an enumeration of the sets $\mathbb{A}_1$ and $\mathbb{B}_1$, respectively) such that

$$\Phi : \mathbb{A}_1 \to \left\{1,2,\dots,\prod_{k=1}^{n} N_k\right\} : \Phi(\boldsymbol{i}) = i_1 + (i_2 - 1)N_1 + \cdots + (i_n - 1)N_{n-1}N_{n-2}\cdots N_1,$$
$$\hat{\Phi} : \mathbb{B}_1 \to \left\{1,2,\dots,\prod_{k=1}^{n} M_k\right\} : \hat{\Phi}(\boldsymbol{j}) = j_1 + (j_2 - 1)M_1 + \cdots + (j_n - 1)M_{n-1}M_{n-2}\cdots M_1.$$

We define the $\prod_{k=1}^{n} N_k \times \prod_{k=1}^{n} N_k$ stochastic matrix P by

$$p_{v,\mu} = \begin{cases} \frac{1}{\gamma_v}, & \text{if } I_{\Phi^{-1}(v)} \subseteq J_{\mathscr{J}(\Phi^{-1}(\mu))} \\ 0, & \text{otherwise,} \end{cases} \tag{3.1}$$

where γ_v is the number of non-zero elements of its v-th row. Consequently, the connection matrix C is defined as in Section 2 and the connection vector $V = \{v_1, v_2, \ldots, v_{N_1 \cdots N_n}\}$ as follows: $v_v = \hat{\Phi}(\mathscr{J}(\Phi^{-1}(v)))$, $v = 1, 2, \ldots, N_1 \cdots N_n$.

Next, we consider $\prod_{k=1}^{n} N_k$ mappings of the form

$$W_{\boldsymbol{i}} : I_{\boldsymbol{i}} \times \mathbb{R} \to J_{\mathscr{J}(\boldsymbol{i})} \times \mathbb{R} : W_{\boldsymbol{i}} \begin{pmatrix} \boldsymbol{x} \\ z \end{pmatrix} = \begin{pmatrix} \boldsymbol{T}_{\boldsymbol{i}}(\boldsymbol{x}) \\ F_{\boldsymbol{i}}(\boldsymbol{x}, z) \end{pmatrix} = \begin{pmatrix} \boldsymbol{T}_{\boldsymbol{i}}(\boldsymbol{x}) \\ s_{\boldsymbol{i}} z + Q_{\boldsymbol{i}}(\boldsymbol{x}) \end{pmatrix}, \tag{3.2}$$

with

$$\boldsymbol{T}_{\boldsymbol{i}}(\boldsymbol{x}) = \begin{pmatrix} T_{1,\boldsymbol{i}}(x_1) \\ T_{2,\boldsymbol{i}}(x_2) \\ \vdots \\ T_{n,\boldsymbol{i}}(x_n) \end{pmatrix} = \begin{pmatrix} a_{1,\boldsymbol{i}} x_1 + b_{1,\boldsymbol{i}} \\ a_{2,\boldsymbol{i}} x_2 + b_{2,\boldsymbol{i}} \\ \vdots \\ a_{n,\boldsymbol{i}} x_n + b_{m,\boldsymbol{i}} \end{pmatrix},$$

for all $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$, $Q_{\boldsymbol{i}}$ a Lipschitz continuous function on $[0,1]^n$, and $s_{\boldsymbol{i}} \in (-1,1)$, for all $\boldsymbol{i} \in \mathbb{A}_1$. The parameters $s_{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{A}_1$ are called vertical scaling factors. We confine the map $W_{\boldsymbol{i}}$ so that it maps the interpolation points that lie on the vertices of $J_{\mathscr{J}(\boldsymbol{i})}$ to the interpolation points that lie on the vertices of $I_{\boldsymbol{i}}$. Hence, we obtain the following relations:

$$T_{k,\boldsymbol{i}}(\hat{x}_{k,j_k-1}) = x_{k,i_k-1}, \quad T_{k,\boldsymbol{i}}(\hat{x}_{k,j_k}) = x_{k,i_k}, \quad k = 1, 2, \ldots, n$$

and

$$F_{\boldsymbol{i}}\Big(\big(\hat{\boldsymbol{x}}_{\mathscr{J}(\boldsymbol{i}) - \sum_{\lambda=1}^{n} \delta_\lambda \boldsymbol{e}_{n,\lambda}}, \hat{z}_{\mathscr{J}(\boldsymbol{i}) - \sum_{\lambda=1}^{n} \delta_\lambda \boldsymbol{e}_{n,\lambda}}\big)\Big) = z_{\boldsymbol{i} - \sum_{\lambda=1}^{n} \delta_\lambda \boldsymbol{e}_{n,\lambda}},$$

for all $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in \{0,1\}^n$.

It is easy to show that there exists a metric ρ_θ (equivalent with the Euclidean metric) such that $W_{\boldsymbol{i}}$ is a contraction for all $\boldsymbol{i} \in \mathbb{A}_1$. To this end, consider the metric ρ_1 defined on $[0,1]^n$ as follows:

$$\rho_1(\boldsymbol{x}, \boldsymbol{y}) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|,$$

and the metric

$$\rho_\theta((\boldsymbol{x}, z), (\boldsymbol{y}, z')) = \rho_1(\boldsymbol{x}, \boldsymbol{y}) + \theta |z - z'|$$

defined on $[0,1]^n \times \mathbb{R}$, where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ and θ is specified below. Then,

$$\begin{aligned} \rho_\theta(W_{\boldsymbol{i}}(\boldsymbol{x}, z), W_{\boldsymbol{i}}(\boldsymbol{y}, z')) &\leqslant \rho_1(\boldsymbol{T}_{\boldsymbol{i}}(\boldsymbol{x}), \boldsymbol{T}_{\boldsymbol{i}}(\boldsymbol{y})) + \theta |s_{\boldsymbol{i}}||z - z'| + \theta |Q_{\boldsymbol{i}}(\boldsymbol{x}) - Q_{\boldsymbol{i}}(\boldsymbol{y})| \\ &\leqslant \bar{a} \rho_1(\boldsymbol{x}, \boldsymbol{y})| + \theta |s_{\boldsymbol{i}}||z - z'| + \theta c \rho_1(\boldsymbol{x}, \boldsymbol{y}) \\ &\leqslant (\bar{a} + \theta c) \rho_1(\boldsymbol{x}, \boldsymbol{y}) + \theta |s_{\boldsymbol{i}}||z - z'|, \end{aligned}$$

where $\bar{a} = \max\{a_{k,\boldsymbol{i}};\ k = 1, 2, \ldots, n,\ \boldsymbol{i} \in \mathbb{A}_1\}$ and c such that $|Q_{\boldsymbol{i}}(\boldsymbol{x}) - Q_{\boldsymbol{i}}(\boldsymbol{y})| \leqslant c\rho_1(\boldsymbol{x}, \boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ and $\boldsymbol{i} \in \mathbb{A}_1$. If we choose $\theta > 0$ such that $\bar{a} + c\theta < 1$, then

$$\rho_\theta(W_{\boldsymbol{i}}(\boldsymbol{x}, z), W_{\boldsymbol{i}}(\boldsymbol{y}, z')) \leqslant q \cdot \rho_\theta((\boldsymbol{x}, z), (\boldsymbol{y}, z')),$$

where $q = \max\{\bar{a} + \theta c, |s_{\boldsymbol{i}}|; \boldsymbol{i} \in \mathbb{A}_1\}$. This means that $W_{\boldsymbol{i}}$ is a contraction with respect to the metric ρ_θ. Thus, $\boldsymbol{W}$ (as defined in Section 2) is also a contraction with respect to the metric $\tilde{h}$. Finally, we define the diagonal matrix $S = \mathrm{diag}(s_{\Phi^{-1}(1)}, s_{\Phi^{-1}(2)}, \ldots, s_{\Phi^{-1}(N)})$, where $N = \prod_{k=1}^n N_k$ and the vector $\boldsymbol{s} = (s_{\Phi^{-1}(1)}, s_{\Phi^{-1}(2)}, \ldots, s_{\Phi^{-1}(N)})$.

As stated above, the RIFS $\{[0,1]^n \times \mathbb{R}, W_{\boldsymbol{i}}, P; \boldsymbol{i} \in \mathbb{A}_1\}$ has a unique attractor G. In general, G is a compact subset of $\mathbb{R}^{n+1}$ containing the points of Δ. The following proposition gives conditions so that G is the graph of a continuous function f. These conditions involve points that lie on $\partial I_{\boldsymbol{i}} \times \mathbb{R}$, for all $\boldsymbol{i} \in \mathbb{A}_1$, (where $\partial I_{\boldsymbol{i}}$ is the boundary of $I_{\boldsymbol{i}}$).

Proposition 3.1 *Let $h \in C([0,1]^n)$ be a Lipschitz function that interpolates the points of Δ (i.e. $h(\boldsymbol{x_i}) = z_{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{A}_0$). If the RIFS defined above satisfies the conditions*

$$F_{\boldsymbol{i}}\left(proj_\lambda \boldsymbol{x} + \hat{x}_{\lambda, j_\lambda} \boldsymbol{e}_{n,\lambda}, h(proj_\lambda \boldsymbol{x} + \hat{x}_{\lambda, j_\lambda} \boldsymbol{e}_{n,\lambda})\right) = h\left(proj_\lambda \boldsymbol{T_i}(\boldsymbol{x}) + x_{\lambda, i_\lambda} \boldsymbol{e}_{n,\lambda}\right), \tag{3.3}$$
$$F_{\boldsymbol{i}}\left(proj_\lambda \boldsymbol{x} + \hat{x}_{\lambda, j_\lambda - 1} \boldsymbol{e}_{n,\lambda}, h(proj_\lambda \boldsymbol{x} + \hat{x}_{\lambda, j_\lambda - 1} \boldsymbol{e}_{n,\lambda})\right) = h\left(proj_\lambda \boldsymbol{T_i}(\boldsymbol{x}) + x_{\lambda, i_\lambda - 1} \boldsymbol{e}_{n,\lambda}\right), \tag{3.4}$$

where $\boldsymbol{j} = \mathscr{J}(\boldsymbol{i})$, for all $\boldsymbol{x} \in J_{\mathscr{J}(\boldsymbol{i})}$, $\boldsymbol{i} = (i_1, i_2, \ldots, i_n) \in \mathbb{A}_1$, $\lambda = 1, 2, \ldots, n$, then its attractor G is the graph of a continuous function f that interpolates the data points. Moreover, f satisfies the functional relation

$$f(\boldsymbol{x}) = F_{\boldsymbol{i}}\left(\boldsymbol{T_i}^{-1}(\boldsymbol{x}), f\left(\boldsymbol{T_i}^{-1}(\boldsymbol{x})\right)\right), \tag{3.5}$$

for all $\boldsymbol{x} \in I_{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{A}_1$.

Proof Let $\langle C([0,1]^n), \|\cdot\|_\infty\rangle$ be the complete metric space of the continuous functions defined on $[0,1]^n$, where

$$\|g\|_\infty = \max\{|g(\boldsymbol{x})|,\ \boldsymbol{x} \in [0,1]^n\}.$$

The set $\mathscr{F} = \{g \in C([0,1]^n) : g \text{ satisfies (3.3)–(3.4)}\}$ is a non-empty ($h \in \mathscr{F}$) complete metric subspace. We define the Read-Bajraktarevic operator $\mathscr{T} : \mathscr{F} \to \mathscr{F}$ by

$$\mathscr{T} g(\boldsymbol{x}) = F_{\boldsymbol{i}}\left(\boldsymbol{T_i}^{-1}(\boldsymbol{x}), g\left(\boldsymbol{T_i}^{-1}(\boldsymbol{x})\right)\right),$$

if $\boldsymbol{x} \in I_{\boldsymbol{i}}$. In view of (3.3)–(3.4), $\mathscr{T} g$ is well defined at $[0,1]^n$. In addition, it is easy to verify that $\mathscr{T}$ is a contraction in $\mathscr{F}$. Hence, $\mathscr{T}$ possesses a unique fixed point $f \in \mathscr{F}$, such that $\mathscr{T} f = f$. With a little bit of effort, we may deduce that the graph of f is the attractor of the RIFS $\{[0,1]^n \times \mathbb{R}, W_{\boldsymbol{i}}, P; \boldsymbol{i} \in \mathbb{A}_1\}$ (see also [4, 8]). $\square$

We refer to functions whose graph arises as attractor of a RIFS, which satisfies the conditions of the above proposition, as FIFs.

The relations (3.3)–(3.4) define a functional system that consists of $2 \cdot n \cdot \prod_{k=1}^n N_k$ equations, which associate $F_{\boldsymbol{i}}$ with h (only at points of $\partial I_{\boldsymbol{i}}$). Considering that $F_{\boldsymbol{i}}(\boldsymbol{x}, z) = s_{\boldsymbol{i}} z +$

$Q_{\boldsymbol{i}}(\boldsymbol{x})$, we obtain the system:

$$Q_{\boldsymbol{i}}\left(\mathrm{proj}_\lambda \boldsymbol{x} + \hat{x}_{\lambda,j_\lambda}\boldsymbol{e}_{n,\lambda}\right) = h\left(\mathrm{proj}_\lambda \boldsymbol{T}_{\boldsymbol{i}}(\boldsymbol{x}) + x_{\lambda,i_\lambda}\boldsymbol{e}_{n,\lambda}\right) - s_{\boldsymbol{i}} \cdot h(\mathrm{proj}_\lambda \boldsymbol{x} + \hat{x}_{\lambda,j_\lambda}\boldsymbol{e}_{n,\lambda}) \tag{3.6}$$

$$Q_{\boldsymbol{i}}\left(\mathrm{proj}_\lambda \boldsymbol{x} + \hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n,\lambda}\right) = h\left(\mathrm{proj}_\lambda \boldsymbol{T}_{\boldsymbol{i}}(\boldsymbol{x}) + x_{\lambda,i_\lambda-1}\boldsymbol{e}_{n,\lambda}\right) - s_{\boldsymbol{i}} \cdot h(\mathrm{proj}_\lambda \boldsymbol{x} + \hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n,\lambda}), \tag{3.7}$$

for all $\boldsymbol{x} \in J_{\mathscr{J}(\boldsymbol{i})}$, $\boldsymbol{i} = (i_1, i_2, \ldots, i_n) \in \mathbb{A}_1$, $\boldsymbol{j} = \mathscr{J}(\boldsymbol{i})$, $\lambda = 1, 2, \ldots, n$, where $s_{\boldsymbol{i}}$ are free parameters.

Remark 3.1 In Proposition 3.1, we stated that h must satisfy a Lipschitz condition. This is necessary because otherwise $Q_{\boldsymbol{i}}$ will not satisfy a Lipschitz condition either.

The following corollary generalises in $\mathbb{R}^n$ an example given in [1] for $n = 1$.

Corollary 3.1 *Let $h \in C([0,1]^n)$ be a Lipschitz function that interpolates the points of Δ. Consider the case*

$$Q_{\boldsymbol{i}}(\boldsymbol{x}) = H(\boldsymbol{T}_{\boldsymbol{i}}(\boldsymbol{x})) - s_{\boldsymbol{i}} \cdot B(\boldsymbol{x}), \quad \textit{for all } \boldsymbol{x} \in J_{\mathscr{J}(\boldsymbol{i})},\ \boldsymbol{i} \in \mathbb{A}_1, \tag{3.8}$$

where H, B are Lipschitz functions defined on $[0,1]^n$ such that

$$H(\boldsymbol{x}) = h(\boldsymbol{x}), \quad \textit{for } \boldsymbol{x} \in \partial I_{\boldsymbol{i}},$$
$$B(\boldsymbol{x}) = h(\boldsymbol{x}), \quad \textit{for } \boldsymbol{x} \in \partial J_{\mathscr{J}(\boldsymbol{i})}.$$

The unique attractor G of the corresponding RIFS $\{[0,1]^n \times \mathbb{R}, W_{\boldsymbol{i}}, P\,; \boldsymbol{i} \in \mathbb{A}_1\}$ is the graph of a continuous function f that interpolates the points of Δ and satisfies

$$f(\boldsymbol{x}) = H(\boldsymbol{x}) + s_{\boldsymbol{i}}(f - B) \circ \boldsymbol{T}_{\boldsymbol{i}}^{-1}(\boldsymbol{x}), \quad \textit{for all } \boldsymbol{x} \in I_{\boldsymbol{i}}, \tag{3.9}$$

$\boldsymbol{i} \in \mathbb{A}_1$.

Proof The proof is straightforward. We may easily confirm that the conditions (3.3)–(3.4) are satisfied. □

Remark 3.2 The FIF f associated with the RIFS of Corollary 3.1 satisfies the inequality

$$\|f - H\| \leqslant \frac{|\boldsymbol{s}|_\infty}{1 - |\boldsymbol{s}|_\infty}\|H - B\|_\infty, \tag{3.10}$$

where $|\boldsymbol{s}|_\infty = \max\{|s_{\boldsymbol{i}}|, \boldsymbol{i} \in \mathbb{A}_1\}$, as we may easily conclude using relation (3.9).

4 Some special cases

In Section 3, we demonstrated that from any multivariate interpolant h one may construct FIFs in more than one way. However, we did not present any specific method that gives the maps of the RIFS, if one chooses a specific interpolant h. Here, we address this problem by presenting two distinct classes of constructions.

Let $\tilde{C}_\Delta$ be the set of all the continuous functions defined on $\cup_{i\in\mathbb{A}_1}\partial I_i$, which interpolate the points of Δ. Using Proposition 3.1, we may proceed with the following construction. We choose $h \in \tilde{C}_\Delta$ a priori to be a Lipschitz function, which interpolates the points of Δ and define Q_i from (3.6)–(3.7) on $\cup_{i\in\mathbb{A}_1}\partial I_i$. Then Q_i are Lipschitz functions (on $\partial J_{\mathscr{J}(i)}$) with constants at most $2L$ for all $\boldsymbol{i} \in \mathbb{A}_1$, where L is the Lipschitz constant of h. Consequently, Q_i may be extended to $J_{\mathscr{J}(i)}$ without increasing its Lipschitz constant (see for example [11], p. 145). Thus, the RIFS consists of contractions, and therefore it has a unique attractor G as stated above. Hence, using Proposition 3.1, we deduce that G is the graph of a continuous function that interpolates the data. However, the extension mentioned above is not trivial. Note that Proposition 3.1 involves only the points of ∂I_i, $\boldsymbol{i} \in \mathbb{A}_1$, and therefore the values of h on the interior of I_i are not taken into account. For that reason, we took h to be in $\tilde{C}_\Delta$. Equivalently, one may assume that h consists of interpolants of $\mathbb{R}^{n-1}$ (i.e. functions that interpolate the points of the set $\Delta_{\kappa,\lambda}$ that consists of the points of Δ that their κ-th ordinate is fixed and equal with $x_{\kappa,\lambda}$, for $\lambda = 1, 2, \ldots, N_\kappa$, $\kappa = 1, 2, \ldots, n$). This becomes more comprehensible in the examples given later.

On the other hand, for the second class, one chooses a function $H \in C([0, 1]^n)$ that interpolates Δ and a function $B \in C([0, 1]^n)$ that interpolates $\hat{\Delta}$ such that $B|_{J_j} = H|_{J_j}$, for all $\boldsymbol{j} \in \mathbb{B}_1$. Using Corollary 3.1, a FIF is easily constructed.

Below, we give some special cases that generalise well-known constructions on $\mathbb{R}$ and some others that are completely new. The first two cases use functions h defined only on $\cup_{i\in\mathbb{A}_1} I_i$, while the third case uses functions that are defined on $[0, 1]^n$.

4.1 Construction I: A generalisation of affine FIFs

Let $h \in \tilde{C}_\Delta$, then we may deduce a generalisation of affine FIFs as follows. We consider the special case where Q_i are of the following form:

$$Q_{\boldsymbol{i}}(\boldsymbol{x}) = \sum_{k=1}^{n} r_{k,\boldsymbol{i}}(\mathrm{proj}_{-k}\boldsymbol{x})x_k + \sum_{k=1}^{n} q_{k,\boldsymbol{i}}(\mathrm{proj}_{-k}\boldsymbol{x}), \tag{4.1}$$

for all $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in J_{\mathscr{J}(i)}$, where $r_{k,\boldsymbol{i}}, q_{k,\boldsymbol{i}} : \mathrm{proj}_{-k}(J_{\mathscr{J}(i)}) \to I_i$ are arbitrary continuous functions such that

$$\begin{aligned} r_{k,\boldsymbol{i}}\big(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda}\boldsymbol{e}_{n-1,\lambda}\big) &= r_{k,\boldsymbol{i}}\big(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n-1,\lambda}\big) = 0, \\ q_{k,\boldsymbol{i}}\big(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda}\boldsymbol{e}_{n-1,\lambda}\big) &= q_{k,\boldsymbol{i}}\big(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n-1,\lambda}\big) = 0, \end{aligned}$$

for all $\boldsymbol{i} \in \mathbb{A}_1$, $\boldsymbol{x} \in J_{\mathscr{J}(i)}$, $\lambda = 1, 2, \ldots, n-1$, $k = \lambda+1, \lambda+2, \ldots, n$. We show that, under the above constraints, if we choose h a priori as a Lipschitz function, then using relations (3.6)–(3.7) we may compute Q_i in terms of the values of $h|_{\partial J_{\mathscr{J}(i)}}$ and the vertical scaling factors s_i, $\boldsymbol{i} \in \mathbb{A}_1$, in a unique way.

Evidently, for $\lambda = 1$, the system (3.6)–(3.7) produces the two equations:

$$\begin{aligned} s_{\boldsymbol{i}} h(\mathrm{proj}_1\boldsymbol{x} + \hat{x}_{1,j_1}\boldsymbol{e}_{n,1}) + r_{1,\boldsymbol{i}}(\mathrm{proj}_{-1}\boldsymbol{x})\hat{x}_{1,j_1} + q_{1,\boldsymbol{i}}(\mathrm{proj}_{-1}\boldsymbol{x}) &= h\left(\mathrm{proj}_1 \boldsymbol{T}(\boldsymbol{x}) + x_{1,i_1}\boldsymbol{e}_{n,1}\right), \\ s_{\boldsymbol{i}} h(\mathrm{proj}_1\boldsymbol{x} + \hat{x}_{1,j_1-1}\boldsymbol{e}_{n,1}) + r_{1,\boldsymbol{i}}(\mathrm{proj}_{-1}\boldsymbol{x})\hat{x}_{1,j_1-1} + q_{1,\boldsymbol{i}}(\mathrm{proj}_{-1}\boldsymbol{x}) &= h\left(\mathrm{proj}_1 \boldsymbol{T}(\boldsymbol{x}) + x_{1,i_1-1}\boldsymbol{e}_{n,1}\right), \end{aligned}$$

for all $\boldsymbol{i} \in \mathbb{A}_1$. Therefore,

$$\begin{aligned} r_{1,\boldsymbol{i}}(\mathrm{proj}_{-1}\boldsymbol{x}) = {} & \frac{h\big(\mathrm{proj}_1 \boldsymbol{T}(\boldsymbol{x}) + x_{1,i_1}\boldsymbol{e}_{n,1}\big) - h\big(\mathrm{proj}_1 \boldsymbol{T}(\boldsymbol{x}) + x_{1,i_1-1}\boldsymbol{e}_{n,1}\big)}{\hat{x}_{1,j_1} - \hat{x}_{1,j_1-1}} \\ & - s_{\boldsymbol{i}} \frac{h\big(\mathrm{proj}_1\boldsymbol{x} + \hat{x}_{1,j_1}\boldsymbol{e}_{n,1}\big) - h\big(\mathrm{proj}_1\boldsymbol{x} + \hat{x}_{1,j_1-1}\boldsymbol{e}_{n,1}\big)}{\hat{x}_{1,j_1} - \hat{x}_{1,j_1-1}}, \\ q_{1,\boldsymbol{i}}(\mathrm{proj}_{-1}\boldsymbol{x}) = {} & h(\mathrm{proj}_1 \boldsymbol{T}(\boldsymbol{x}) + x_{1,i_1-1}\boldsymbol{e}_{n,1}) - s_{\boldsymbol{i}} h(\mathrm{proj}_1 \boldsymbol{T}(\boldsymbol{x}) + \hat{x}_{1,j_1-1}\boldsymbol{e}_{n-,1}) \\ & - r_{1,\boldsymbol{i}}(\mathrm{proj}_{-1}\boldsymbol{x})\hat{x}_{1,j_1-1}, \end{aligned}$$

for all $\boldsymbol{x} \in J_{\mathscr{J}(\boldsymbol{i})}$.

In general, for $\lambda \geqslant 2$, we have

$$\begin{aligned} & s_{\boldsymbol{i}} h(\mathrm{proj}_\lambda \boldsymbol{x} + \hat{x}_{\lambda,j_\lambda}\boldsymbol{e}_{n,\lambda}) + \sum_{k=1}^{\lambda-1} r_{k,\boldsymbol{i}}(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda}\boldsymbol{e}_{n,\lambda})x_k + r_{\lambda,\boldsymbol{i}}(\mathrm{proj}_{-\lambda}\boldsymbol{x})\hat{x}_{\lambda,j_\lambda} \\ & \quad + \sum_{k=1}^{\lambda-1} q_{k,\boldsymbol{i}}(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda}\boldsymbol{e}_{n-1,\lambda}) + q_{\lambda,\boldsymbol{i}}(\mathrm{proj}_{-\lambda}\boldsymbol{x}) = h(\mathrm{proj}_\lambda T(\boldsymbol{x}) + x_{\lambda,i_\lambda}\boldsymbol{e}_{n,\lambda}) \end{aligned}$$

and

$$\begin{aligned} & s_{\boldsymbol{i}} h\big(\mathrm{proj}_\lambda \boldsymbol{x} + \hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n,\lambda}\big) + \sum_{k=1}^{\lambda-1} r_{k,\boldsymbol{i}}(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n-1,\lambda})x_k + r_{\lambda,\boldsymbol{i}}(\mathrm{proj}_{-\lambda}\boldsymbol{x})\hat{x}_{\lambda,j_\lambda-1} \\ & \quad + \sum_{k=1}^{\lambda-1} q_{k,\boldsymbol{i}}\big(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n-1,\lambda}\big) + q_{\lambda,\boldsymbol{i}}(\mathrm{proj}_{-\lambda}\boldsymbol{x}) = h\big(\mathrm{proj}_\lambda T(\boldsymbol{x}) + x_{\lambda,i_\lambda-1}\boldsymbol{e}_{n,\lambda}\big). \end{aligned}$$

Hence,

$$\begin{aligned} r_{\lambda,\boldsymbol{i}}(\mathrm{proj}_{-\lambda}\boldsymbol{x}) = {} & \frac{h\left(\mathrm{proj}_\lambda \boldsymbol{T}(\boldsymbol{x}) + x_{\lambda,i_\lambda}\boldsymbol{e}_{n,\lambda}\right) - h\left(\mathrm{proj}_\lambda \boldsymbol{T}(\boldsymbol{x}) + x_{\lambda,i_\lambda-1}\boldsymbol{e}_{n,\lambda}\right)}{\hat{x}_{\lambda,j_\lambda} - \hat{x}_{\lambda,j_\lambda-1}} \\ & - s_{\boldsymbol{i}} \frac{h(\mathrm{proj}_\lambda\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda}\boldsymbol{e}_{n,\lambda}) - h(\mathrm{proj}_\lambda\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n,\lambda})}{\hat{x}_{\lambda,j_\lambda} - \hat{x}_{\lambda,j_\lambda-1}} \\ & - \sum_{k=1}^{\lambda-1} \frac{r_{k,\boldsymbol{i}}(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda}\boldsymbol{e}_{n-1,\lambda}) - r_{k,\boldsymbol{i}}(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n-1,\lambda})}{\hat{x}_{\lambda,j_\lambda} - \hat{x}_{\lambda,j_\lambda-1}} x_k \\ & - \sum_{k=1}^{\lambda-1} \frac{q_{k,\boldsymbol{i}}(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda}\boldsymbol{e}_{n-1,\lambda}) - q_{k,\boldsymbol{i}}(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n-1,\lambda})}{\hat{x}_{\lambda,j_\lambda} - \hat{x}_{\lambda,j_\lambda-1}} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} q_{\lambda,\boldsymbol{i}}(\mathrm{proj}_{-\lambda}\boldsymbol{x}) = {} & h(\mathrm{proj}_\lambda T_{\boldsymbol{i}}(\boldsymbol{x}) + x_{\lambda,i_\lambda-1}\boldsymbol{e}_{n,\lambda}) - s_{\boldsymbol{i}} h(\mathrm{proj}_\lambda\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n,\lambda}) \\ & - \sum_{k=1}^{\lambda-1} r_{k,\boldsymbol{i}}(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n-1,\lambda})x_k - r_{\lambda,\boldsymbol{i}}(\mathrm{proj}_{-\lambda}\boldsymbol{x})\hat{x}_{\lambda,j_\lambda-1} \\ & - \sum_{k=1}^{\lambda-1} q_{k,\boldsymbol{i}}(\mathrm{proj}_{-k,\lambda}\boldsymbol{x} + \hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n-1,\lambda}), \end{aligned} \tag{4.3}$$

for all $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{i} \in \mathbb{A}_1$.

The functions $r_{\lambda,\boldsymbol{i}}$, $q_{\lambda,\boldsymbol{i}}$ are Lipschitz for all $\boldsymbol{i} \in \mathbb{A}_1$, $\lambda = 1, 2, \ldots, n$, since h is Lipschitz. Hence, $Q_{\boldsymbol{i}}$ is also Lipschitz, for $\boldsymbol{i} \in \mathbb{A}_1$. Therefore, according to Proposition 3.1, the attractor of the RIFS is the graph of a continuous function $f : [0,1]^n \to \mathbb{R}$. We emphasise that the functions $r_{\lambda,\boldsymbol{i}}$, $q_{\lambda,\boldsymbol{i}}$ (hence the function Q_i) are determined only from the values of h on $\cup_{\boldsymbol{i}\in\mathbb{A}_1} \partial I_{\boldsymbol{i}}$.

One may select h as any multivariate interpolant. One possible selection that we use in our examples follows. Let $\boldsymbol{i} \in \mathbb{A}_1$ and consider the multivariate function of the form

$$u_{\boldsymbol{i}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\{0,1\}^n} \alpha_{\boldsymbol{i},\boldsymbol{k}} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}, \tag{4.4}$$

where $\boldsymbol{k} = (k_1, k_2, \ldots, k_n) \in \{0,1\}^n$, for all $\boldsymbol{x} \in I_{\boldsymbol{i}}$. For example, for $n = 1$, $u_{\boldsymbol{i}}$ is the affine function $u_{\boldsymbol{i}}(x) = \alpha x + \beta$ and for $n = 2$, $u_{\boldsymbol{i}}$ is the bivariate function $u_{\boldsymbol{i}}(x,y) = \alpha_1 x + \alpha_2 y + \alpha_3 xy + \alpha_4$. If we presume that $u_{\boldsymbol{i}}$ interpolates the points of Δ, whose projections on $[0,1]^n$ are the vertices of $I_{\boldsymbol{i}}$, then we obtain a linear system

$$u_{\boldsymbol{i}}\left(\boldsymbol{x}_{\boldsymbol{i}-\sum_{\lambda=1}^n \delta_\lambda e_{n,\lambda}}\right) = z_{\boldsymbol{i}-\sum_{\lambda=1}^n \delta_\lambda e_{n,\lambda}},$$

for all $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in \{0,1\}^n$, which consists of 2^n equations. This system can always be solved for the map's parameters $\alpha_{\boldsymbol{i},\boldsymbol{k}}$, $\boldsymbol{k} \in \{0,1\}^n$. One possible selection of h is

$$h|_{I_{\boldsymbol{i}}} = u_{\boldsymbol{i}}|_{I_{\boldsymbol{i}}}.$$

Keep in mind, however, that only the values of h on the boundaries of $I_{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathscr{A}_1$ are taken into account.

Example 1 For $n = 1$, relation (4.1) takes the form:

$$Q_i(x) = r_i x + q_i,$$

for $i = 1, 2, \ldots, N_1$, $x \in J_{\mathscr{J}(i)}$. This selection gives the well-known piecewise self-affine FIF, which has been extensively studied [3, 18]. In this case, h is defined only on $\{x_0, x_1, \ldots, x_N\}$, therefore only the values $z_0 = h(x_0)$, $z_1 = h(x_1), \ldots,$ and $z_N = h(x_N)$ are taken into account. In Figure 2, two FIFs, which correspond to the same selection of Δ and $\hat{\Delta}$, are shown.

Example 2 For $n = 2$, we obtain

$$\begin{aligned}
\Delta &= \{(x_{i_1}, y_{i_2}, z_{i_1,i_2}) \in I \times \mathbb{R}; i_k = 0, 1, \ldots, N_k,\ k = 1, 2\},\\
\hat{\Delta} &= \{(\hat{x}_{j_1}, \hat{y}_{j_2}, \hat{z}_{j_1,j_2}) \in I \times \mathbb{R}; j_k = 0, 1, \ldots, M_k,\ k = 1, 2\},\\
\boldsymbol{i} &= (i_1, i_2) \in \{0, 1, \ldots, N_1\} \times \{0, 1, \ldots, N_2\} = \mathbb{A}_0,\\
\boldsymbol{j} &= (j_1, j_2) \in \{0, 1, \ldots, M_1\} \times \{0, 1, \ldots, M_2\} = \mathbb{B}_0,\\
\mathbb{A}_1 &= \{1, 2, \ldots, N_1\} \times \{1, 2, \ldots, N_2\},\\
\mathbb{B}_1 &= \{1, 2, \ldots, M_1\} \times \{1, 2, \ldots, M_2\}.
\end{aligned}$$

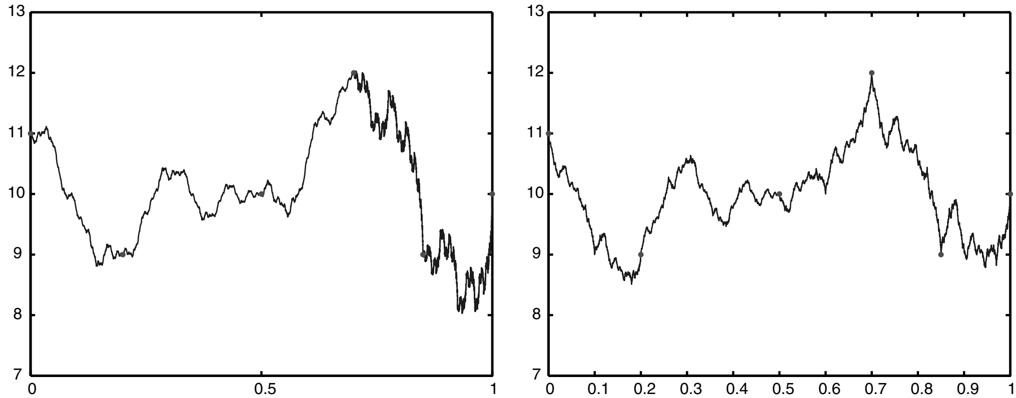

FIGURE 2. The two FIFs shown above interpolate the points of the same set Δ (consisting of six points). The difference is due to the selection of two distinct stochastic matrices.

In this case, relation (4.1) takes the form

$$Q_{\boldsymbol{i}}(x,y) = r_{1,\boldsymbol{i}}(y)x + r_{2,\boldsymbol{i}}(x)y + q_{1,\boldsymbol{i}}(y) + q_{2,\boldsymbol{i}}(x),$$

for $\boldsymbol{i} \in \mathbb{A}_1$, $(x,y) \in J_{\mathscr{J}(i)}$, where

$$\begin{aligned}
r_{1,\boldsymbol{i}}(y) &= \frac{h(x_{i_1}, T_{2,\boldsymbol{i}}(y)) - h(x_{i_1-1}, T_{2,\boldsymbol{i}}(y))}{\hat{x}_{j_1} - \hat{x}_{j_1-1}} - s_{\boldsymbol{i}}\frac{h(\hat{x}_{j_1}, y) - h(\hat{x}_{j_1-1}, y)}{\hat{x}_{j_1} - \hat{x}_{j_1-1}},\\
q_{1,\boldsymbol{i}}(y) &= h(x_{i_1-1}, T_{2,\boldsymbol{i}}(y)) - s_{\boldsymbol{i}} h(\hat{x}_{j_1-1}, y) - r_{1,\boldsymbol{i}}(y)\hat{x}_{j_1-1},\\
r_{2,\boldsymbol{i}}(x) &= \frac{h(T_{1,\boldsymbol{i}}(x), y_{i_2}) - h(T_{1,\boldsymbol{i}}(x), y_{i_2-1})}{\hat{y}_{j_2} - \hat{y}_{j_2-1}} - s_{\boldsymbol{i}}\frac{h(x, \hat{y}_{j_2}) - h(x, \hat{y}_{j_2-1})}{\hat{y}_{j_2} - \hat{y}_{j_2-1}}\\
&\quad - \frac{r_{1,\boldsymbol{i}}(\hat{y}_{j_2}) - r_{1,\boldsymbol{i}}(\hat{y}_{j_2-1})}{\hat{y}_{j_2} - \hat{y}_{j_2-1}}x - \frac{q_{1,\boldsymbol{i}}(\hat{y}_{j_2}) - q_{1,\boldsymbol{i}}(\hat{y}_{j_2-1})}{\hat{y}_{j_2} - \hat{y}_{j_2-1}},\\
q_{2,\boldsymbol{i}}(x) &= h(T_{1,\boldsymbol{i}}(x), y_{i_2-1}) - s_{\boldsymbol{i}} h(x, \hat{y}_{j_2-1}) - r_{1,\boldsymbol{i}}(\hat{y}_{j_2-1})x - r_{2,\boldsymbol{i}}(x)\hat{y}_{j_2-1} - q_{1,\boldsymbol{i}}(\hat{y}_{j_2-1}).
\end{aligned}$$

The above equations define a RIFS on $\mathbb{R}^3$ that has a unique attractor G, which is the graph of a continuous function $f : [0,1]^2 \to \mathbb{R}$. We refer to an attractor of this nature as FIS. Clearly, the construction of f depends on the selection of h. One might say that h consists of one-dimensional interpolants, that connect the points of Δ (Figures 3–5). If we fix $\Delta, \hat{\Delta}$, then for any Lipschitz function $h \in \tilde{C}_{\Delta}$ that interpolates Δ and for any selection of vertical scaling factors $\boldsymbol{s}$, we can construct a FIS $f_{h,\boldsymbol{s}}$ that also interpolates Δ.

If we choose h as in (4.4), then we are led to a major generalisation of [6]. In this case, the graph of h on $\partial I_{\boldsymbol{i}}$ is a closed polygonal line that connects the four vertices of Δ, whose projections are the vertices of $\partial I_{\boldsymbol{i}}$. In fact, if the interpolation points that lie inside $\Delta \cap [\hat{x}_{j_1-1}, \hat{x}_{j_1}] \times \{\hat{y}_{j''}\} \times \mathbb{R}$ or $\Delta \cap \{\hat{x}_{j'}\} \times [\hat{y}_{j_2-1}, \hat{y}_{j_2}] \times \mathbb{R}$ are collinear, for all $\boldsymbol{j} = (j_1, j_2) \in \mathbb{B}_1$, $(j', j'') \in \mathbb{B}_0$, then it is easy to show that the above construction is identical to the one presented in [6]. Indeed, in this case, $h(x_{i_1}, T_{2,\boldsymbol{i}}(y))$ and $h(x_{i_1-1}, T_{2,\boldsymbol{i}}(y))$ are affine functions of y and $h(T_{1,\boldsymbol{i}}(x), y_{i_2})$ and $h(T_{1,\boldsymbol{i}}(x), y_{i_2-1})$ are affine functions of x.

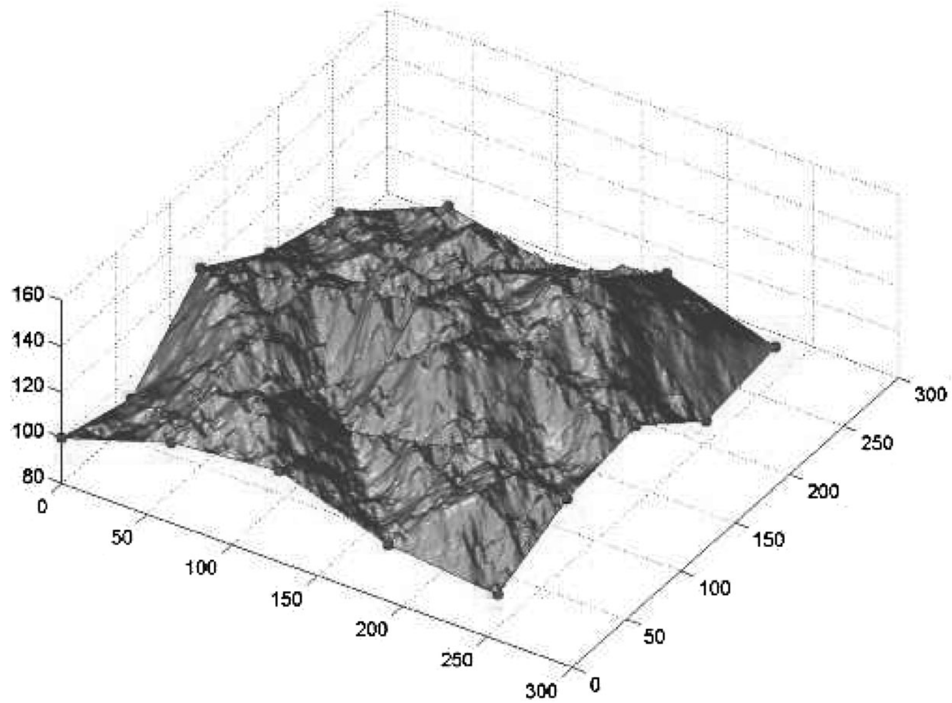

FIGURE 3. A FIS that interpolates 5×5 interpolation points (see Table 1). The values of the interpolation points are shown with dots. The values of h at ∂I_i, $\boldsymbol{i} \in \mathbb{A}_1$, (i.e. the one-dimensional interpolants) are polygonal lines and they are shown in light grey.

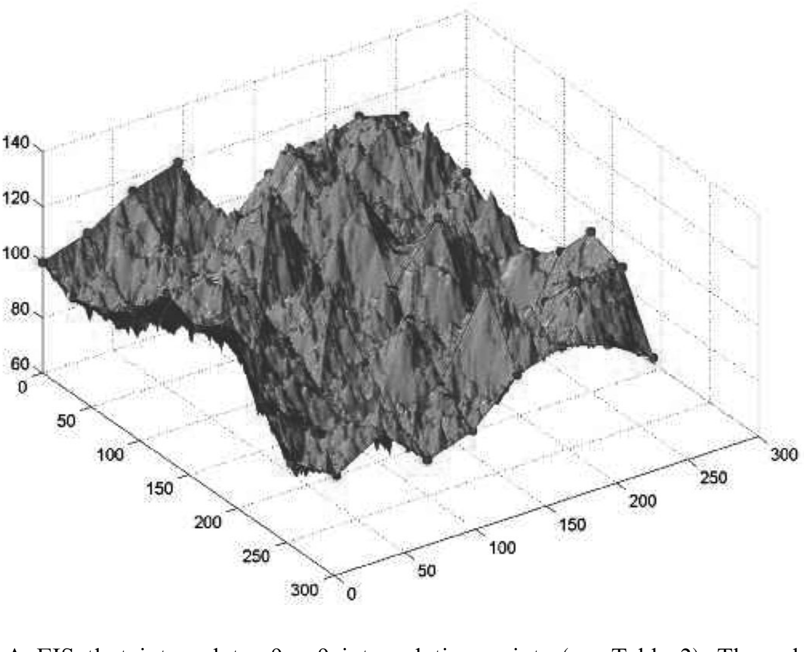

FIGURE 4. A FIS that interpolates 9×9 interpolation points (see Table 2). The values of the interpolation points are shown with dots. The values of h at ∂I_i, $\boldsymbol{i} \in \mathbb{A}_1$, (i.e. the one-dimensional interpolants) are polygonal lines and they are shown in light grey.

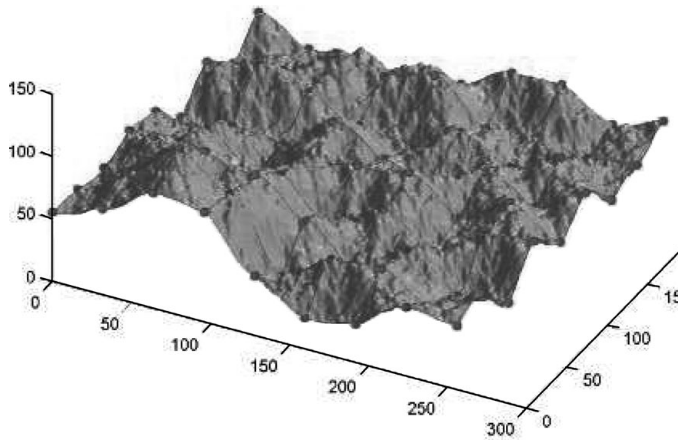

FIGURE 5. A FIS that interpolates 9×9 interpolation points. The values of the interpolation points are shown with dots. The values of h at $\partial I_{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{A}_1$, (i.e. the one-dimensional interpolants) are hermite interpolation functions (see also Table 3).

Hence, $r_{1,\boldsymbol{i}}(y)$, $r_{2,\boldsymbol{i}}(x)$, $q_{1,\boldsymbol{i}}(y)$ and $q_{2,\boldsymbol{i}}(x)$ are also affine functions, and $Q_{\boldsymbol{i}}$ takes the form

$$Q_{\boldsymbol{i}}(x) = a_0 + a_1 x + a_2 y + a_3 xy.$$

Using equations (3.6)–(3.7), we deduce that

$$\begin{aligned} Q_{\boldsymbol{i}}(\hat{x}_{j_1-1}, \hat{y}_{j_2-1}) &= h(x_{i_1-1}, y_{i_2-1}) - s_{\boldsymbol{i}} \cdot h(\hat{x}_{j_1-1}, \hat{y}_{j_2-1}) = z_{i_1-1,i_2-1} - s_{\boldsymbol{i}} \hat{z}_{j_1-1,j_2-1}, \\ Q_{\boldsymbol{i}}(\hat{x}_{j_1-1}, \hat{y}_{j_2}) &= h(x_{i_1-1}, y_{i_2}) - s_{\boldsymbol{i}} \cdot h(\hat{x}_{j_1-1}, \hat{y}_{j_2}) = z_{i_1-1,i_2} - s_{\boldsymbol{i}} \hat{z}_{j_1-1,j_2}, \\ Q_{\boldsymbol{i}}(\hat{x}_{j_1}, \hat{y}_{j_2-1}) &= h(x_{i_1}, y_{i_2-1}) - s_{\boldsymbol{i}} \cdot h(\hat{x}_{j_1}, \hat{y}_{j_2-1}) = z_{i_1,i_2-1} - s_{\boldsymbol{i}} \hat{z}_{j_1,j_2-1}, \\ Q_{\boldsymbol{i}}(\hat{x}_{j_1}, \hat{y}_{j_2}) &= h(x_{i_1}, y_{i_2}) - s_{\boldsymbol{i}} \cdot h(\hat{x}_{j_1}, \hat{y}_{j_2}) = z_{i_1,i_2} - s_{\boldsymbol{i}} \hat{z}_{j_1,j_2}. \end{aligned}$$

By the uniqueness of the solution of the above system, we have the result. Figures 3 and 4 show FISs defined on arbitrary interpolation points using this form of h (see also tables 1, 2).

In Figure 5, h is chosen such that its intersection with each one of the planes $x = x_i$, $y = y_j$, $i = 0, 1, \ldots, N_1$, $j = 0, 1, 2, \ldots, N_2$ are Hermite-type polynomials (of degree 3), which interpolate the corresponding points of Δ. For this purpose, we added to the interpolation points arbitrary values for the partial derivatives of h as shown in Table 3. One may choose h such that its intersections with the planes given above are splines (of any type) or any other interpolant.

4.2 Construction II: Using partial differential equations

Next, we work with interpolants $h \in \tilde{C}^k_{\Delta'}$ (i.e. functions h that are defined only on $\cup_{\boldsymbol{i} \in \mathbb{A}_1} \partial I_{\boldsymbol{i}}$ and have continuous partial derivatives up to k-th order). Let $\mathscr{R}([0,1]^n)$ be the set containing all the subsets of $[0,1]^n$ of the form

$$[a_1, b_1] \times [a_2, b_2] \times, \ldots, \times [a_n, b_n]$$

Table 1. *The interpolation points, the vertical scaling factors and the connection vector used for the RIFS shown in Figure 3. The points of* $\hat{\Delta}$ *are shown with bold characters.*

Δ						
		x				
		0	$\frac{256}{4}$	$2\frac{256}{4}$	$3\frac{256}{4}$	256
y	0	**100**	090	**120**	100	**090**
	$\frac{256}{6}$	115	130	130	100	110
	$2\frac{256}{6}$	**120**	110	**140**	126	**100**
	$3\frac{256}{6}$	105	120	130	140	115
	256	**100**	115	**120**	095	**100**

$$S = \begin{pmatrix} 0.45 & 0.55 & -0.5 & 0.35 \\ 0.5 & -0.85 & 0.75 & -0.75 \\ -0.45 & 0.85 & -0.45 & 0.5 \\ -0.65 & -0.55 & 0.45 & 0.25 \end{pmatrix}$$

$$V = (2, 1, 4, 3, 4, 1, 3, 2, 1, 4, 3, 2, 2, 1, 3, 4)$$

Table 2. *The interpolation points, the vertical scaling factors and the connection vector used for the RIFS shown in Figure 4. The points of* $\hat{\Delta}$ *are shown with bold characters.*

Δ										
		x								
		0	$\frac{256}{8}$	$2\frac{256}{8}$	$3\frac{256}{6}$	$4\frac{256}{8}$	$5\frac{256}{8}$	$6\frac{256}{8}$	$7\frac{256}{8}$	256
y	0	**100**	105	**115**	120	**090**	105	**110**	115	**110**
	$\frac{256}{8}$	095	080	105	070	100	120	105	120	105
	$2\frac{256}{8}$	**098**	095	**100**	115	**110**	105	**095**	095	**105**
	$3\frac{256}{8}$	105	110	105	130	100	120	095	090	095
	$4\frac{256}{8}$	**120**	115	**100**	095	**080**	095	**115**	095	**090**
	$5\frac{256}{8}$	115	120	100	110	090	070	090	070	100
	$6\frac{256}{8}$	**125**	100	**090**	095	**105**	090	**085**	095	**115**
	$7\frac{256}{8}$	105	090	080	080	090	120	090	110	110
	256	**090**	080	**090**	075	**080**	095	**100**	095	**085**

$$S = \begin{pmatrix} +0.55 & +0.55 & -0.50 & +0.65 & +0.45 & -0.40 & -0.47 & +0.32 \\ +0.50 & -0.85 & +0.75 & -0.55 & -0.40 & +0.34 & -0.41 & +0.65 \\ -0.55 & +0.45 & -0.45 & +0.50 & -0.74 & +0.72 & -0.57 & -0.43 \\ -0.95 & -0.55 & +0.75 & -0.65 & +0.52 & -0.35 & +0.53 & -0.45 \\ +0.65 & +0.55 & -0.60 & +0.85 & +0.65 & -0.40 & -0.67 & +0.32 \\ +0.50 & -0.45 & +0.55 & -0.75 & -0.60 & +0.34 & -0.61 & +0.45 \\ -0.55 & +0.45 & -0.65 & +0.50 & -0.54 & +0.75 & -0.47 & -0.43 \\ -0.65 & -0.55 & +0.75 & -0.65 & +0.52 & -0.75 & +0.63 & -0.25 \end{pmatrix}$$

$$\begin{aligned} V = (&1, 5, 10, 4, 8, 12, 5, 2, 7, 11, 8, 15, 16, 3, 9, 6, \\ &10, 11, 9, 5, 12, 5, 8, 1, 4, 9, 10, 16, 11, 3, 8, 2, \\ &4, 7, 1, 10, 15, 12, 6, 15, 1, 5, 10, 4, 8, 12, 5, 2, \\ &7, 11, 8, 15, 16, 3, 9, 6, 10, 11, 9, 5, 12, 5, 8, 1) \end{aligned}$$

and a k-th order partial differential equation (PDE) defined on the set $R \in \mathscr{R}([0,1]^n)$ such that its solution g satisfies

$$g|_{\partial R} = v, \tag{4.5}$$

where $v \in C^k(\partial R)$. Assuming that the PDE is uniquely solvable for any $R \in \mathscr{R}([0,1]^n)$ and $v \in C^k(\partial R)$ and that the solution is a Lipschitz function, we consider the

Table 3. *The interpolation points used for the RIFS shown in Figure 5. The points of $\hat{\Delta}$ are shown with bold characters. To construct h on the borders of the grid, we used Hermite-type interpolation polynomials. The vertical scaling factors and the connection vector are the same as in Table 2.*

Δ

y \ x	0	$\frac{256}{8}$	$2\frac{256}{8}$	$3\frac{256}{6}$	$4\frac{256}{8}$	$5\frac{256}{8}$	$6\frac{256}{8}$	$7\frac{256}{8}$	256
0	**100**	105	**115**	120	**090**	105	**110**	115	**110**
$\frac{256}{8}$	095	080	105	070	100	120	105	120	105
$2\frac{256}{8}$	**098**	095	**100**	115	**110**	105	**095**	095	**105**
$3\frac{256}{8}$	105	110	105	130	100	120	095	090	095
$4\frac{256}{8}$	**120**	115	**100**	095	**080**	095	**115**	095	**090**
$5\frac{256}{8}$	115	120	100	110	090	070	090	070	100
$6\frac{256}{8}$	**125**	100	**090**	095	**105**	090	**085**	095	**115**
$7\frac{256}{8}$	105	090	080	080	090	120	090	110	110
256	**090**	080	**090**	075	**080**	095	**100**	095	**085**

$\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}$

y \ x	0	$\frac{256}{8}$	$2\frac{256}{8}$	$3\frac{256}{6}$	$4\frac{256}{8}$	$5\frac{256}{8}$	$6\frac{256}{8}$	$7\frac{256}{8}$	256
0	−20, 10	−50, −20	50, 50	30, 40	−50, −30	20, 30	50, −60	−10, 40	10, 20
$\frac{256}{8}$	10, 10	−20, 10	50, −60	40, 40	−30, 20	30, 50	−60, 20	40, −50	20, −10
$2\frac{256}{8}$	50, 10	20, 10	50, −20	−10, 50	10, 40	10, −60	−20, −40	50, 60	40, 50
$3\frac{256}{8}$	−60, −20	−40, 50	60, 40	−50, −60	20, 40	50, 60	−10, 50	−10, 20	10, 50
$4\frac{256}{8}$	−60, 10	40, 20	20, 50	50, 40	−20, −30	50, 30	−10, −60	10, 40	10, 20
$5\frac{256}{8}$	20, −40	−50, 20	40, 50	−60, 20	40, 50	60, −10	50, 10	20, 10	50, −20
$6\frac{256}{8}$	10, 50	10, 40	10, −60	−20, 40	50, −60	40, 50	−30, 20	−30, −50	−60, −10
$7\frac{256}{8}$	−40, 10	20, −50,	−50, −60	20, 40	50, 20	10, −50	10, 20	10, 50	−20, −10
256	50, −10	−40, 10	60, −20	40, 50	−60, 40	50, −60	−20, 40	50, −60	−10, 50

operator

$$\mathscr{P}_R : C^k(\partial R) \to C^k(R) : \mathscr{P}_R(v) = g,$$

that assigns any function v defined on ∂R to the solution g of the corresponding PDE with boundary conditions as in (4.5). We study the case where $Q_i(\boldsymbol{x}) = H(\boldsymbol{T}_i(\boldsymbol{x})) - s_i \cdot B(\boldsymbol{x})$, for all $\boldsymbol{x} \in J_{\mathscr{J}(\boldsymbol{i})}, \boldsymbol{i} \in \mathbb{A}_1$, as in Corollary 3.1, where

$$H|_{I_i} = \mathscr{P}_{I_i}\left(h|_{\partial I_i}\right),$$
$$B|_{\mathscr{J}(\boldsymbol{i})} = \mathscr{P}_{J_{\mathscr{J}(i)}}\left(h|_{\partial J_{\mathscr{J}(i)}}\right),$$

for all $\boldsymbol{i} \in \mathbb{A}_1$. In this case, the conditions of Corollary 3.1 are satisfied, thence the unique attractor G of the corresponding RIFS is the graph of a continuous function f that interpolates the points of Δ.

Example 1 We may choose the Laplace PDE

$$\frac{\partial^2 g}{\partial x_1^2} + \frac{\partial^2 g}{\partial x_2^2} + \cdots + \frac{\partial^2 g}{\partial x_n^2} = 0$$

that has been extensively studied. It is interesting to see that if the interpolation points that lie inside $\Delta \cap [\hat{x}_{k,j_k-1}, \hat{x}_{k,j_k}] \times \mathbb{R}$ are collinear for all $\boldsymbol{j} = (j_1, \ldots, j_n) \in \mathbb{B}_1$, $k = 1, \ldots, n$, and we choose h as in (4.4), then we may easily show that for $n = 1$, the above construction gives the well-known affine FIFs and for $n = 2$ the construction is identical to the one described in Example 2 of Section 4.1, in the case where h has the same form. Evidently, for any solvable PDE with boundary conditions as above, we obtain a different fractal surface. The choices are limitless.

Consider now the operator

$$\mathscr{D}_{\mathscr{P},\Delta',\hat{\Delta}',P,s} : \tilde{C}^k_{\Delta'} \to C([0,1]^n) : \mathscr{D}_{\mathscr{P},\Delta',\hat{\Delta}',P,s}(h) = f,$$

which assigns a FIF f to any function h defined on $\cup_{\boldsymbol{i} \in \mathbb{A}_1} \partial I_{\boldsymbol{i}}$, that has all the partial derivatives of order k and interpolates the points of Δ. The function f is the attractor of a RIFS defined as in Corollary 3.1, where $H|_{I_i} = \mathscr{P}_{I_i}(h|_{\partial I_i}), B|_{\mathscr{J}(\boldsymbol{i})} = \mathscr{P}_{J_{\mathscr{J}(i)}}(h|_{\partial \mathscr{J}(\boldsymbol{i})})$, thence

$$f|_{I_i} = \mathscr{D}_{\mathscr{P},\Delta',\hat{\Delta}',P,\boldsymbol{s}}(h)|_{I_i} = \mathscr{P}_{I_i}(h) + s_{\boldsymbol{i}} \left(f|_{J_{\mathscr{J}i}} - \mathscr{P}_{J_{\mathscr{J}(i)}}(h) \right) \circ \boldsymbol{T}_{\boldsymbol{i}}^{-1}, \tag{4.6}$$

for all $\boldsymbol{i} \in \mathbb{A}_1$. We use the notation $\mathscr{P}_{I_i}(h) = \mathscr{P}_{I_i}(h|_{\partial I_i})$, $\mathscr{P}_{J_{\mathscr{J}(i)}}(h) = \mathscr{P}_{\partial J_{\mathscr{J}(i)}}(h|_{\mathscr{J}(\boldsymbol{i})})$ to shorten the proof of the following proposition.

Proposition 4.1 *If for any $R \in \mathbb{R}([0,1]^n)$ the operator $\mathscr{P}_R$ is linear and injective, then $\mathscr{D}_{\mathscr{P},\Delta',\hat{\Delta}',P,S}$ is also a linear and injective operator.*

Proof We may prove the linearity as follows: Consider $h_1, h_2 \in C^k_{\Delta'}$. Then, for all $\boldsymbol{x} \in I_{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{A}_1$, we have

$$f_1(\boldsymbol{x}) = \mathscr{D}_{\mathscr{P},\Delta',\hat{\Delta}',P,\boldsymbol{s}}(h_1)(\boldsymbol{x}) = \mathscr{P}_{I_i}(h_1)(\boldsymbol{x}) + s_{\boldsymbol{i}} \left(f_1 - \mathscr{P}_{J_{\mathscr{J}(i)}}(h_1) \right) \circ \boldsymbol{T}_{\boldsymbol{i}}^{-1}(\boldsymbol{x}),$$
$$f_2(\boldsymbol{x}) = \mathscr{D}_{\mathscr{P},\Delta',\hat{\Delta}',P,\boldsymbol{s}}(h_2)(\boldsymbol{x}) = \mathscr{P}_{I_i}(h_2)(\boldsymbol{x}) + s_{\boldsymbol{i}} \left(f_2 - \mathscr{P}_{J_{\mathscr{J}(i)}}(h_2) \right) \circ \boldsymbol{T}_{\boldsymbol{i}}^{-1}(\boldsymbol{x}).$$

Therefore, for any $\lambda, \mu \in \mathbb{R}$, $\boldsymbol{i} \in \mathbb{A}_1$, from the uniqueness of the solution of functional equation (3.9), we have

$$\begin{aligned} \lambda f_1|_{I_i} + \mu f_2|_{I_i} &= \lambda \mathscr{D}_{\mathscr{P},\Delta',\hat{\Delta}',P,\boldsymbol{s}}(h_1)|_{I_i} + \mu \mathscr{D}_{\mathscr{P},\Delta',\hat{\Delta}',P,\boldsymbol{s}}(h_2)|_{I_i} \\ &= \lambda \mathscr{P}_{I_i}(h_1) + \lambda s_{\boldsymbol{i}} \left(f_1|_{J_{\mathscr{J}(i)}} - \mathscr{P}_{J_{\mathscr{J}(i)}}(h_1) \right) \circ \boldsymbol{T}_{\boldsymbol{i}}^{-1} \\ &\quad + \mu \mathscr{P}_{I_i}(h_2) + \mu s_{\boldsymbol{i}} \left(f_2|_{J_{\mathscr{J}(i)}} - \mathscr{P}_{J_{\mathscr{J}(i)}}(h_2) \right) \circ \boldsymbol{T}_{\boldsymbol{i}}^{-1} \\ &= \mathscr{P}_{I_i}(\lambda h_1 + \mu h_2) + s_{\boldsymbol{i}} \left((\lambda f_1 + \mu f_2)|_{J_{\mathscr{J}(i)}} - \mathscr{P}_{J_{\mathscr{J}(i)}}(\lambda h_1 + \mu h_2) \right) \circ \boldsymbol{T}_{\boldsymbol{i}}^{-1} \\ &= \mathscr{D}_{\mathscr{P},\Delta',\hat{\Delta}',P,\boldsymbol{s}}(\lambda h_1 + \mu h_2)|_{I_i}. \end{aligned}$$

Evidently, from the conditions of Corollary 3.1, if $\mathscr{D}_{\mathscr{P},\Delta',\hat{\Delta}',P,s}(h) = 0$, then $h = 0$. Therefore, the operator is injective. □

For example, the Laplace PDE satisfies the conditions of the proposition.

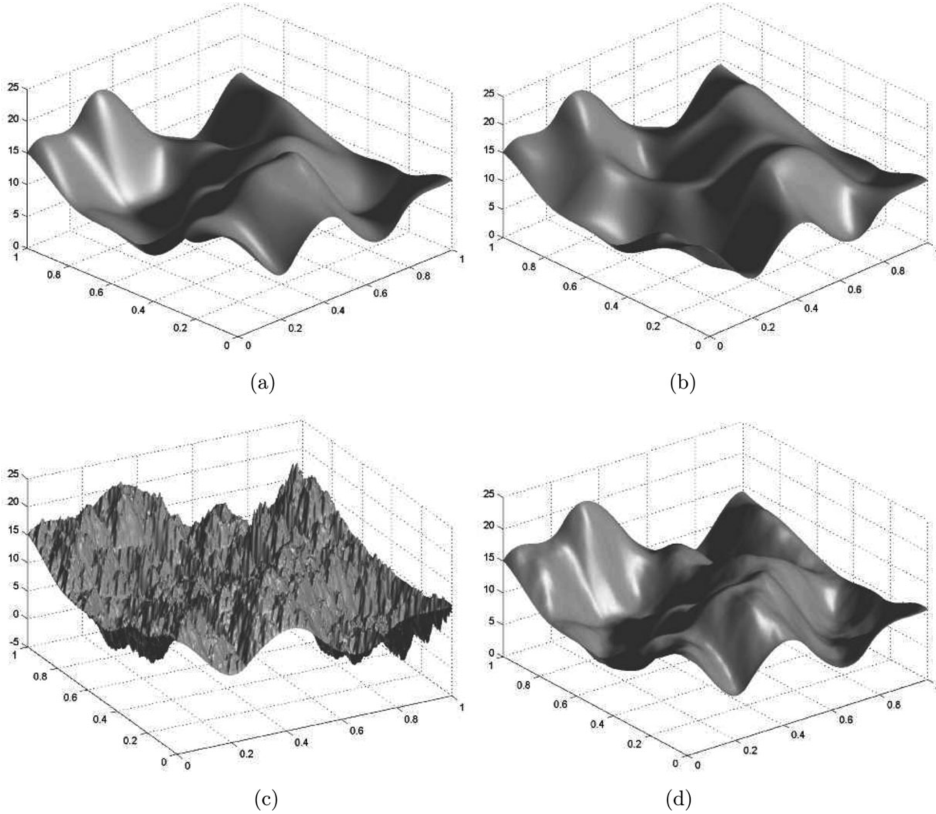

FIGURE 6. (a), (b) Two multivariate C^1 interpolation functions H and B that satisfy the conditions of Corollary 3.1 and Corollary 6.1. (c) The attractor of the corresponding RIFS using arbitrary vertical scaling factors. (d) The attractor of the corresponding RIFS using vertical scaling factors that satisfy the conditions of Corollary 6.1 is a C^1 function.

4.3 Construction III: Using multivariate interpolants

One may use arbitrary multivariate interpolants H and B that satisfy the conditions of Corollary 3.1 (as in Figure 6). Below, we give a construction that is based only on one interpolant. Consider $\mathscr{B} : C([0,1]^n) \to C([0,1]^n)$ to be a linear operator such that $\mathscr{B}(g)|_{\partial J_j} = g|_{\partial J_j}$, for $\boldsymbol{j} \in \mathbb{B}_1$. For example, if $c_{\boldsymbol{j}} : J_{\boldsymbol{j}} \to J_{\boldsymbol{j}}$ are functions (other than the identity function) such that $c_{\boldsymbol{j}}(\boldsymbol{x}) = \boldsymbol{x}$, for all $\boldsymbol{x} \in \partial J_{\boldsymbol{j}}$, $\boldsymbol{j} \in \mathbb{B}_1$, then we may choose $\mathscr{B}$ such that $\mathscr{B}(g)|_{J(\boldsymbol{j})} = g \circ c_{\boldsymbol{j}}$. Let $h \in C([0,1])$ be any multivariate interpolant of Δ (e.g. a spline). If we set $H = h$ and $B = \mathscr{B}(h)$, then we may define the operator $D_{\Delta',\hat{\Delta}',P,\boldsymbol{s}}$, which assigns a FIF to any function h that interpolates the points of Δ according to Corollary 3.1. We may prove the linearity of the operator as in equation (4.2). In [21], a similar construction, which generalises the Fourier approximations, is presented for $n = 1$.

5 The box-counting dimension of a class of FIFs

Let B be any non-empty compact subset of $\mathbb{R}^{n+1}$ and let $\mathscr{N}(B, \varepsilon)$ be the *smallest number of (closed) balls of radius* ε that cover B. Let

$$\underline{D} = \underline{D}(B) = \liminf_{\varepsilon > 0} \frac{\log \mathscr{N}(B,\varepsilon)}{\log(1/\varepsilon)} \quad \text{and} \quad \overline{D} = \overline{D}(B) = \limsup_{\varepsilon > 0} \frac{\log \mathscr{N}(B,\varepsilon)}{\log(1/\varepsilon)}$$

be the *lower* and *upper box-counting dimension* of B, respectively; if

$$D = D(B) = \lim_{\varepsilon \to 0} \frac{\log \mathscr{N}(B,\varepsilon)}{\log(1/\varepsilon)}$$

exists, then D is called the *box-counting dimension* of B [2]. In the latter case, we use the notation $D = D(B) = \underline{D}(B) = \overline{D}(B)$ and say 'B has box-counting dimension D'. In practice, we usually use covers of closed cubes of side length $(1/k^v)$, where $k \in \mathbb{N}$. If $\mathscr{N}_v(B)$ denotes the smallest number of cubes of side $1/k^v$ that cover B and

$$D = \lim_{v \to \infty} \frac{\log \mathscr{N}_v(B)}{\log k^v}$$

exists, then B has box-counting dimension D. However, to compute $D(B)$, we often use covers that differ from those above. Assume that one uses covers from the set $\{\mathscr{C}_\varepsilon : \varepsilon > 0\}$, which is formed by *sets of diameter* ε, and let $\mathscr{N}'(\varepsilon)$ be the minimum number of sets in $\mathscr{C}_\varepsilon$ that cover B. If we can find constant numbers c_1 and c_2 such that $c_1\mathscr{N}(B,\varepsilon) \leqslant \mathscr{N}'(\varepsilon) \leqslant c_2\mathscr{N}(B,\varepsilon)$, then $\mathscr{N}(B,\varepsilon)$ can be replaced by $\mathscr{N}'(\varepsilon)$ in the computation of $D(B)$ (the proof is straightforward).

Henceforth, we consider the FIF described in the first case of Section 4. The functions $Q_{\boldsymbol{i}}$ are of the form

$$Q_{\boldsymbol{i}}(\boldsymbol{x}) = \sum_{k=1}^{n} r_{k,\boldsymbol{i}}(\mathrm{proj}_{-k}\boldsymbol{x})x_k + \sum_{k=1}^{n} q_{k,\boldsymbol{i}}(\mathrm{proj}_{-k}\boldsymbol{x}),$$

where $r_{k,\boldsymbol{i}}, q_{k,\boldsymbol{i}}$ are defined by (4.2)–(4.3) for all $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in J_{\mathscr{J}(i)}$, $\boldsymbol{i} \in \mathbb{A}_1$. We compute the box-counting dimension of the graph G of this FIF in the case

$$h = u|_{\cup_{\boldsymbol{i}\in\mathbb{A}_1} \partial I_{\boldsymbol{i}}},$$

where u is a multivariate function of the form (4.4).

Definition 5.1 The maximum range of a continuous function g inside the rectangle $I \subset \mathbb{R}^n$ is defined by

$$\mathscr{R}_g(I) = \max\{|g(\boldsymbol{x}) - g(\boldsymbol{y})| : \boldsymbol{x}, \boldsymbol{y} \in I\}.$$

Lemma 5.1 *Let K be a rectangle of $\mathbb{R}^n$ and W a map of the form*

$$W : K \times \mathbb{R} \to K \times \mathbb{R} : W\begin{pmatrix} \boldsymbol{x} \\ z \end{pmatrix} = \begin{pmatrix} \boldsymbol{T}(\boldsymbol{x}) \\ F(\boldsymbol{x}, z) \end{pmatrix} = \begin{pmatrix} \boldsymbol{T}(\boldsymbol{x}) \\ sz + Q(\boldsymbol{x}) \end{pmatrix},$$

where Q is a Lipschitz function and $\boldsymbol{T}$ an affine function. Then for any continuous function $g : K \to \mathbb{R}$, there is $L > 0$ such that

$$\mathscr{R}_{F\circ(\boldsymbol{T}^{-1}, g\circ \boldsymbol{T}^{-1})}(\boldsymbol{T}(K)) \leqslant |s|\mathscr{R}_g(K) + L \cdot d(K), \tag{5.1}$$

where $d(K)$ is the diameter of K.

Proof The proof is straightforward. Since Q is a Lipschitz function, we have

$$\begin{aligned}|F(\boldsymbol{x},z)-F(\boldsymbol{x}',z')| &\leqslant |s||z-z'|+|Q(\boldsymbol{x})-Q(\boldsymbol{x}')|\\ &\leqslant |s||z-z'|+L\|\boldsymbol{x}-\boldsymbol{x}'\|,\end{aligned}$$

for all $\boldsymbol{x},\boldsymbol{x}'\in K$. □

Theorem 5.1 *If the points of Δ', $\hat{\Delta}'$ are equidistant*

$$\begin{aligned}x_{k,i_k+1}-x_{k,i_k}&=\delta,\\ \hat{x}_{k,j_k+1}-\hat{x}_{k,j_k}&=\psi,\end{aligned}$$

for all $\boldsymbol{i}\in\mathbb{A}_1$, $\boldsymbol{j}\in\mathbb{B}_1$, where $\delta,\psi\in\mathbb{R}$, $\alpha=\frac{\psi}{\delta}\in\mathbb{N}$, $k=1,2,\ldots,n$, where $N_k=N$, $M_k=M$, the connection matrix C of the RIFS is irreducible and there exists a $\boldsymbol{j}\in\mathbb{B}_1$ such that at least one of the points of $\Delta\cap(J_{\boldsymbol{j}}\times\mathbb{R})$ does not lie on the unique multivariate surface of the form (4.4), which is defined by the points of $\Delta\cap(\partial J_{\boldsymbol{j}}\times\mathbb{R})$, then the box-counting dimension of the graph is

$$D(G)=\begin{cases}1+\log_\alpha\lambda, & \text{if } \lambda>\alpha^{n-1}\\ n, & \text{otherwise}\end{cases},$$

where $\lambda=\rho(S\cdot C)$.

Proof The proof follows the one presented in [6]. To compute the box-counting dimension, we use covers of the form

$$\mathscr{C}=\left\{\left[\frac{v_1-1}{\alpha^r},\frac{v_1}{\alpha^r}\right]\times\cdots\times\left[\frac{v_n-1}{\alpha^r},\frac{v_n}{\alpha^r}\right]\times\left[b,b+\frac{1}{\alpha^r}\right]\ :\ v_1,\ldots,v_n,r\in\mathbb{N},\ b\in\mathbb{R}\right\}.$$

$\mathscr{C}$ contains overlapping $1/\alpha^r$-mesh cubes. We define,

$\mathscr{N}^*(r)$ as the minimum number of cubes in $\mathscr{C}$ necessary to cover A and

$\mathscr{N}(r)$ as the smallest number of $\dfrac{1}{\alpha^r}$ -mesh cubes which cover A,

for $r>0$. We may easily prove that $\mathscr{N}(r)\leqslant\mathscr{N}^*(r)\leqslant 2^n\mathscr{N}(r)$, thus $\mathscr{N}(r)$ may be replaced by $\mathscr{N}^*(r)$ in the computation of the dimension.

From the irreducibility of the connection matrix C and the fact that there exists a $\boldsymbol{j}\in\mathbb{B}_1$, such that at least one of the points of $\Delta\cap\left(J_{\boldsymbol{j}}\times\mathbb{R}\right)$ does not lie on the unique multivariate surface of the form (4.4), which is defined by the points of $\Delta\cap\left(\partial J_{\boldsymbol{j}}\times\mathbb{R}\right)$, we deduce that there are points $(\tilde{x_{\boldsymbol{i}}},\tilde{z_{\boldsymbol{i}}})\in G\times(I_{\boldsymbol{i}}\mathbb{R})$, for all $\boldsymbol{i}\in\mathbb{A}_1$, that do not lie on the multivariate surface of the form (4.4), which is defined by the points of $\Delta\cap(\partial I_{\boldsymbol{i}}\times\mathbb{R})$. Let $V_{\boldsymbol{i}}$ denote this vertical distance and $R_{\boldsymbol{i}}$ the range of f inside $I_{\boldsymbol{i}}$. We define the non-negative vectors H_1, U_1 and $\boldsymbol{I}$ by

$$H_1 = \begin{pmatrix} V_{\Phi^{-1}(1)} \\ V_{\Phi^{-1}(2)} \\ \cdot \\ \cdot \\ \cdot \\ V_{\Phi^{-1}(N^n)} \end{pmatrix}, \quad U_1 = \begin{pmatrix} R_{\Phi^{-1}(1)} \\ R_{\Phi^{-1}(2)} \\ \cdot \\ \cdot \\ \cdot \\ R_{\Phi^{-1}(N^n)} \end{pmatrix} \quad \text{and} \quad \boldsymbol{I} = (1, 1, \ldots, 1)^t .$$

Therefore,

$$\Omega\,(H_1 \cdot \alpha^r) - N^n \leqslant \mathcal{N}^*(r) \leqslant \Omega\,(U_1 \cdot \alpha^r + \boldsymbol{I}) \cdot \left(\left[\frac{\alpha^r}{N}\right] + 1\right)^n,$$

where $\Omega(U) = u_1 + u_2 + \cdots + u_{N^n}$, for $U = (u_1, u_2 \ldots, u_{N^n}) \in \mathbb{R}^{N^n}$ and $[\cdot]$ the greatest integer function. After the second iteration, we obtain α^n n-dimensional cubes of side $\frac{1}{\alpha N}$. With the help of Lemma 5.1, we find that the maximum ranges are contained (as coordinates) in the vector

$$U_2 = SC \cdot U_1 + \alpha^2 L \frac{1}{N},$$

while the heights produced are contained (as coordinates) in the vector

$$H_2 = SC \cdot H_1.$$

Thus,

$$\Omega\,(H_2 \cdot \alpha^r) - \alpha^n N^n \leqslant \mathcal{N}^*(r) \leqslant \Omega\,(U_2 \cdot \alpha^r + \alpha^n \boldsymbol{I}) \cdot \left(\left[\frac{\alpha^r}{\alpha N}\right] + 1\right)^n.$$

After κ iterations, where $r - \mu - 1 \leqslant \kappa < r - \mu$, $\mu = \frac{\log N}{\log \alpha} - 1 > 0$, we obtain the main inequality that is used to derive the result. This inequality is a generalisation of inequality (8) of [6]:

$$\Omega(H_\kappa \cdot \alpha^r) - N^n \alpha^{n(k-1)} \leqslant \mathcal{N}^*(r) \leqslant \Omega\left(U_\kappa \alpha^r + \mathbb{I}\alpha^{n(\kappa-1)}\right) \cdot \left(\left[\frac{\alpha^r}{\alpha^{\kappa-1} N}\right] + 1\right)^n,$$

where

$$U_\kappa = SC \cdot U_{\kappa-1} + L \frac{\alpha^{(n-1)\kappa+2-n}}{N}$$

and

$$H_\kappa = SC \cdot H_{\kappa-1}.$$

With the help of the Perron-Frobenius Theorem, we obtain the result (see [6] for details). Similar remarks with the ones that follow the proof of the theorem in [6] are also in effect in $\mathbb{R}^n$. □

For the general case, where the interpolation points are not equidistant, we obtain a result similar to the one presented in [3]. The details of the computation follow closely those of Theorem 4.2 of [3]. The main idea is to derive functional inequalities for $\mathcal{N}(G, \varepsilon)$ that can be used to estimate the behaviour of $\mathcal{N}(G, \varepsilon)$ as ε decreases to zero. Since the proofs of all the following theorems are almost identical to the ones presented in [3], we only present the crucial points.

We first introduce the class of covers that allow us to relate covers of different sizes. For $\epsilon > 0$, define the set

$$\boldsymbol{\tau}_{\varepsilon,\boldsymbol{m}} = \{\boldsymbol{\tau}_{\boldsymbol{l}} = (\tau_{1,l_1}, \tau_{2,l_2}, \ldots, \tau_{n,l_n}) \in \boldsymbol{I}\,; l_k = 0, 1, \ldots, m_k,\ k = 1, 2, \ldots, n\},$$

where $\boldsymbol{l} = (l_1, l_2, \ldots, l_n) \in \mathbb{L} = \{0, 1, \ldots, m_1\} \times \cdots \times \{0, 1, \ldots, m_n\}$, $\boldsymbol{m} = (m_1, \ldots, m_k)$. The set $\boldsymbol{\tau}_{\varepsilon,\boldsymbol{m}}$ is called an ε-partition if

(1) $\tau_{k,l_k} \in (-\frac{\varepsilon}{2}, 1)$,

(2) $\frac{\varepsilon}{2} < \tau_{k,l_k+1} - \tau_{k,l_k} < \varepsilon$

for all $l_k = 0, 1, \ldots, m_k - 1$, $k = 1, 2, \ldots, n$. A cover $\mathscr{C}$ will be called an ε-column cover of G with associated ε-partition $\boldsymbol{\tau}_{\varepsilon,\boldsymbol{m}}$, if there are $\{n_{\boldsymbol{l}};\ \boldsymbol{l} \in \mathbb{L}\} \subset \mathbb{N}$ and $\{\xi_{\boldsymbol{l}};\ \boldsymbol{l} \in \mathbb{L}\} \subset \mathbb{R}$ such that

$$\mathscr{C} = \left\{ \left[\tau_{1,l_1}, \tau_{1,l_1} + \varepsilon\right] \times \cdots \times \left[\tau_{n,l_n}, \tau_{n,l_n} + \varepsilon\right] \times \left[\xi_{\boldsymbol{l}} + (\mu - 1)\varepsilon, \xi_{\boldsymbol{l}} + \mu\varepsilon\right] : \mu = 1, \ldots, n_{\boldsymbol{l}}; \boldsymbol{l} \in \mathbb{L} \right\}.$$

We define $\mathscr{N}^*(\varepsilon) = \min\{|\mathscr{C}| : \mathscr{C}$ is an ε-column cover of $G\}$, where $|\mathscr{C}|$ is the cardinality of $\mathscr{C}$, and $\mathscr{N}(\varepsilon)$ as the minimum number of $(n+1)$-dimensional cubes of side length ε. Then the inequalities

$$\mathscr{N}(\varepsilon) \leqslant \mathscr{N}^*(\varepsilon) \leqslant 2^n \mathscr{N}(\varepsilon) \tag{5.2}$$

show that $\mathscr{N}(\varepsilon)$ can be replaced by $\mathscr{N}^*(\varepsilon)$ in the calculation of $D(G)$. Let $\mathscr{N}_{\boldsymbol{i}}(\varepsilon) = \min\{|\mathscr{C}| : \mathscr{C}$ is an ε-column cover of $G \cap (I_{\boldsymbol{i}} \times \mathbb{R})\}$ for all $\boldsymbol{i} \in \mathbb{A}_1$.

The following Lemma is a generalisation of Lemma 4.2 of [3].

Lemma 5.2 *There exists $P_{\boldsymbol{i}}$, $P'_{\boldsymbol{i}} > 0$, $\boldsymbol{i} \in \mathbb{A}_1$, such that, for $0 < \varepsilon < 1$,*

$$\left|\frac{s_{\boldsymbol{i}}}{\underline{a}_{\boldsymbol{i}}}\right| \sum_{\boldsymbol{i}' : I_{\boldsymbol{i}'} \in J_{\mathscr{J}(\boldsymbol{i})}} \mathscr{N}_{\boldsymbol{i}'}\left(\frac{\varepsilon}{\underline{a}_{\boldsymbol{i}}}\right) - \frac{P_{\boldsymbol{i}}}{\varepsilon^n} \leqslant \mathscr{N}_{\boldsymbol{i}}(\varepsilon) \leqslant \left|\frac{s_{\boldsymbol{i}}}{\underline{a}_{\boldsymbol{i}}}\right| \sum_{\boldsymbol{i}' : I_{\boldsymbol{i}'} \in J_{\mathscr{J}(\boldsymbol{i})}} \mathscr{N}_{\boldsymbol{i}'}\left(\frac{\varepsilon}{\underline{a}_{\boldsymbol{i}}}\right) + \frac{P'_{\boldsymbol{i}}}{\varepsilon^n},$$

where $\underline{a}_{\boldsymbol{i}} = \max\{a_{k,\boldsymbol{i}};\ k = 1, 2, \ldots, n\}$, for all $\boldsymbol{i} \in \mathbb{A}_1$.

Proof If $s_{\boldsymbol{i}} = 0$, the proof is straightforward. We suppose that $s_{\boldsymbol{i}} \neq 0$, then $W_{\boldsymbol{i}}$ is invertible. Let $\mathscr{C}_{\boldsymbol{i}}$ be a minimal ε-column cover of $G \cap (I_{\boldsymbol{i}} \times \mathbb{R})$ and let R be a typical column in $\mathscr{C}_{\boldsymbol{i}}$, which consists of v $(n+1)$-dimensional cubes of side length ε. The map $W_{\boldsymbol{i}}$ has the form

$$W_{\boldsymbol{i}}\begin{pmatrix}\boldsymbol{x}\\ z\end{pmatrix} = \begin{pmatrix}\boldsymbol{T}_{\boldsymbol{i}}(\boldsymbol{x})\\ F_{\boldsymbol{i}}(\boldsymbol{x})\end{pmatrix} = \begin{pmatrix}\boldsymbol{T}_{\boldsymbol{i}}(\boldsymbol{x})\\ s_{\boldsymbol{i}}z + Q_{\boldsymbol{i}}(\boldsymbol{x})\end{pmatrix},$$

where $Q_{\boldsymbol{i}}|_{I_{\boldsymbol{k}}}$, is a multivariate function of the form (4.4), for all $\boldsymbol{k} : I_{\boldsymbol{k}} \subseteq J_{\mathscr{J}(\boldsymbol{i})}$. In this case, one can easily prove that $W_{\boldsymbol{i}}^{-1}$ is a function of the form

$$W_{\boldsymbol{i}}^{-1}\begin{pmatrix}\boldsymbol{x}\\ z\end{pmatrix} = \begin{pmatrix}\boldsymbol{T}_{\boldsymbol{i}}^{-1}(\boldsymbol{x})\\ \frac{z}{s_{\boldsymbol{i}}} + Q'_{\boldsymbol{i}}(\boldsymbol{x})\end{pmatrix},$$

where $Q_i'|_{T_i(I_k)}$ is also a multivariate function of the form (4.4), for all $k : I_k \subseteq J_{\mathcal{J}(i)}$. Therefore, using Lemma 5.1, there exists $L > 0$ such that $W_i^{-1}(R)$ is inside a column of volume

$$\frac{\varepsilon}{a_{1,i}} \times \cdots \times \frac{\varepsilon}{a_{n,i}} \times \left(\frac{v\varepsilon}{|s_i|} + L\varepsilon \right).$$

Similarly, there exists $L' > 0$, such that $W_i^{-1}(R)$ contains a column of volume

$$\frac{\varepsilon}{a_{1,i}} \times \cdots \times \frac{\varepsilon}{a_{n,i}} \times \left(\frac{v\varepsilon}{|s_i|} - L'\varepsilon \right).$$

This means, for example, that $W_i^{-1}(R)$ can be covered by

$$\frac{v\underline{a}_i}{|s_i|} + L\underline{a}_i + 1$$

$(n+1)$-dimensional cubes of side length $\frac{\varepsilon}{\underline{a}_i}$, and there are at most $2^n \frac{a_i}{\varepsilon^n} + 2^n$ such columns. From this point on, the proof is similar with the proof presented in Lemma 4.2 of [3]. □

Theorem 5.2 *Let f be a FIF defined as above, with irreducible connection matrix C and graph G. Let $S(d) = \operatorname{diag}\{\underline{a}_{\Phi^{-1}(1)}^{d-1}|s_{\Phi^{-1}(1)}|, \underline{a}_{\Phi^{-1}(2)}^{d-1}|s_{\Phi^{-1}(2)}|, \ldots, \underline{a}_{\Phi^{-1}(N)}^{d-1}|s_{\Phi^{-1}(N)}|\}$ and let D be the unique value so that $\rho(S(D)\cdot C) = 1$. If $\rho(S(n)\cdot C) > 1$ and there exist a $\boldsymbol{j} \in \mathbb{B}_1$ such that at least one of the points of $\Delta \cap (J_j \times \mathbb{R})$ does not lie on the unique multivariate surface of the form (4.4), which is defined by the points of $\Delta \cap (\partial J_j \times \mathbb{R})$, then $D(G) = D$, otherwise $D(G) = n$.*

Proof The assumptions of the theorem ensure that $\lim_{\varepsilon\to 0} \varepsilon\mathcal{N}_i^*(\varepsilon) = \infty$, for all $\boldsymbol{i} \in \mathbb{A}_1$. To prove this, we may use the Perron-Frobenius Theorem as in Lemma 4.3 of [3]. The rest of the proof is similar with Theorem 4.2 in [3]. We give only the system of solutions of the functional equalities that are associated with the inequalities of Lemma 5.2:

$$\phi_i(R, \gamma, \varepsilon) = \gamma\epsilon^{-D} + \frac{R}{1-\lambda}\varepsilon^{-n}v_i, \quad \boldsymbol{i} \in A_1.$$ □

6 C^p fractal interpolation functions

In this section, we present methods for the construction of functions of class C^p using RIFS. These functions are called in excess *C^p FIFs* (see [17]) since they are generated through RIFS, in spite of the fact that they are not of fractal nature. Again, the Read-Bajraktarevic operator, defined in Section 3, plays an important role.

Theorem 6.1 *Let $h \in C^p([0,1]^n)$, $p \geqslant 0$, be a function that interpolates the points of Δ, such that all its derivatives of order p satisfy the Lipschitz condition. If the RIFS defined in Section 3 satisfies the conditions*

$$\frac{\partial^d F_i \circ \left(\boldsymbol{T}_i^{-1}, h \circ \boldsymbol{T}_i^{-1}\right)}{\partial x_1^{d_1} \ldots \partial x_n^{d_n}} \left(proj_\lambda \boldsymbol{x} + x_{\lambda, i_\lambda - 1}\boldsymbol{e}_{n,\lambda}\right) = \frac{\partial^d h}{\partial x_1^{d_1} \ldots \partial x_n^{d_n}} \left(proj_\lambda \boldsymbol{x} + x_{\lambda, i_\lambda - 1}\boldsymbol{e}_{n,\lambda}\right), \quad (6.1)$$

$$\frac{\partial^d F_{\boldsymbol{i}} \circ \left(\boldsymbol{T}_{\boldsymbol{i}}^{-1}, h \circ \boldsymbol{T}_{\boldsymbol{i}}^{-1}\right)}{\partial x_1^{d_1} \ldots \partial x_n^{d_n}} \left(proj_\lambda \boldsymbol{x} + x_{\lambda,i_\lambda} \boldsymbol{e}_{n,\lambda}\right) = \frac{\partial^d h}{\partial x_1^{d_1} \ldots \partial x_n^{d_n}} \left(proj_\lambda \boldsymbol{x} + x_{\lambda,i_\lambda} \boldsymbol{e}_{n,\lambda}\right), \tag{6.2}$$

for all $(d_1, d_2, \ldots, d_n) \in \mathbb{N}^n$ *such that* $0 \leqslant \sum_{\kappa=1}^n d_\kappa = d \leqslant p$, $d_\lambda \geqslant 1$, *if* $d > 0$, $\boldsymbol{x} \in \partial I_{\boldsymbol{i}}$, $\boldsymbol{i} = (i_1, i_2, \ldots, i_n) \in \mathbb{A}_1$, $\lambda = 1, 2, \ldots, n$, *the functions* $F_{\boldsymbol{i}}$ *are of class* C^p *and the vertical scaling factors satisfy* $|s_{\boldsymbol{i}}| < \underline{a_{\boldsymbol{i}}}^p$, *where* $\underline{a_{\boldsymbol{i}}} = \min\{a_{\lambda,i};\ \lambda = 1, 2, \ldots, n\}$, *for all* $\boldsymbol{i} \in \mathbb{A}_1$, *then its attractor* G *is the graph of a function* f *of class* C^p *that interpolates the data points.*

Proof Let $\langle C^p([0,1]^n), \|\cdot\|_{C^p}\rangle$ be the complete metric space of the continuous functions defined on $[0,1]^n$ that have continuous derivatives up to order p, where

$$\|g\|_{C^p} = \max\left\{\left|\frac{\partial^d g}{\partial x_1^{d_1} \ldots \partial x_n^{d_n}}(\boldsymbol{x})\right|,\ \boldsymbol{x} \in [0,1]^n, d_\kappa \in \mathbb{N} \text{ such that } \sum_{\kappa=1}^n d_\kappa = d \leqslant p\right\}.$$

The set $\mathscr{F} = \{g \in C^p([0,1]^n) : g \text{ satisfies (6.1)–(6.2)}\}$ is a non-empty ($h \in \mathscr{F}$) complete metric subspace. We define the Read-Bajraktarevic operator $\mathscr{T} : \mathscr{F} \to \mathscr{F}$ by

$$\mathscr{T} g(\boldsymbol{x}) = F_{\boldsymbol{i}}\left(\boldsymbol{T}_{\boldsymbol{i}}^{-1}(\boldsymbol{x}), g\left(\boldsymbol{T}_{\boldsymbol{i}}^{-1}(\boldsymbol{x})\right)\right),$$

if $\boldsymbol{x} \in I_{\boldsymbol{i}}$. In view of (6.1)–(6.2), $\mathscr{T} g$ is well defined in $[0,1]^n$ and maps $\mathscr{F}$ on to itself. We now prove that $\mathscr{T}$ is a contraction with respect to the $\|\cdot\|_{C^p}$ norm. Let $\boldsymbol{i} \in \mathbb{A}_0$, $g_1, g_2 \in \mathscr{F}$. For $\boldsymbol{x} \in I_{\boldsymbol{i}}$, we have

$$\begin{aligned}
\left|\frac{\partial^d(\mathscr{T} g_1 - \mathscr{T} g_2)}{\partial x_1^{d_1} \ldots \partial x_n^{d_n}}(\boldsymbol{x})\right| &= \left|s_{\boldsymbol{i}} \prod_{\kappa=1}^n a_{i_\kappa}^{-d_\kappa}\right| \cdot \left|\frac{\partial^d(g_1 - g_2)}{\partial x_1^{d_1} \ldots \partial x_n^{d_n}}\left(\boldsymbol{T}_{\boldsymbol{i}}^{-1}(\boldsymbol{x})\right)\right| \\
&\leqslant \left|\frac{s_{\boldsymbol{i}}}{\underline{a_{\boldsymbol{i}}}^d}\right| \cdot \left|\frac{\partial^d(g_1 - g_2)}{\partial x_1^{d_1} \ldots \partial x_n^{d_n}}\left(\boldsymbol{T}_{\boldsymbol{i}}^{-1}(\boldsymbol{x})\right)\right|.
\end{aligned}$$

Thus,

$$\|\mathscr{T} g_1|_{I_{\boldsymbol{i}}} - \mathscr{T} g_2|_{I_{\boldsymbol{i}}}\|_{C^p} \leqslant \left|\frac{s_{\boldsymbol{i}}}{\underline{a}_{\boldsymbol{i}}^d}\right| \cdot \|g_1|_{J_{\mathscr{I}(I_{\boldsymbol{i}})}} - g_2|_{J_{\mathscr{I}(I_{\boldsymbol{i}})}}\|_{C^p} \leqslant \left|\frac{\bar{s}}{\underline{a}^d}\right| \cdot \|g_1 - g_2\|,$$

for all $\boldsymbol{i} \in \mathbb{A}_1$, where $\bar{s} = \max\{|s_{\boldsymbol{i}}|, \boldsymbol{i} \in \mathbb{A}_1\}$, $\underline{a} = \min\{a_{\boldsymbol{i}}, \boldsymbol{i} \in \mathbb{A}_1\}$, and therefore $\mathscr{T}$ is a contraction. Hence, $\mathscr{T}$ possesses a unique fixed point $f \in \mathscr{F}$. □

Remark 6.1 The relation $|s_{\boldsymbol{i}}| < \underline{a_{\boldsymbol{i}}}^p$, for all $\boldsymbol{i} \in \mathbb{A}_1$, ensures that the Read-Bajraktarevic operator is a contraction with respect to the $\|\cdot\|_{C^p}$ norm. In [5], Barnsley and Harrington obtain the same relation (for $n = 1$) through integration.

Remark 6.2 Considering that $F_{\boldsymbol{i}}(\boldsymbol{x}, z) = s_{\boldsymbol{i}} z + Q_{\boldsymbol{i}}(\boldsymbol{x})$, the relations (6.1)–(6.2) become

$$\begin{aligned}
\frac{\partial^d Q_{\boldsymbol{i}}}{\partial x_1^{d_1} \ldots \partial x_n^{d_n}}\left(\mathrm{proj}_\lambda \boldsymbol{x} + \hat{x}_{\lambda,j_\lambda - 1} \boldsymbol{e}_{n,\lambda}\right) &= \prod_{\kappa=1}^n a_{i_\kappa}^{d_\kappa} \frac{\partial^d h}{\partial x_1^{d_1} \ldots \partial x_n^{d_n}}\left(\mathrm{proj}_\lambda \boldsymbol{T}(\boldsymbol{x}) + x_{\lambda,i_\lambda - 1} \boldsymbol{e}_{n,\lambda}\right) \\
&\quad - s_{\boldsymbol{i}} \frac{\partial^d h}{\partial x_1^{d_1} \ldots \partial x_n^{d_n}}\left(\mathrm{proj}_\lambda \boldsymbol{x} + \hat{x}_{\lambda,i_\lambda - 1} \boldsymbol{e}_{n,\lambda}\right),
\end{aligned} \tag{6.3}$$

$$\frac{\partial^d Q_{\boldsymbol{i}}}{\partial x_1^{d_1}\dots\partial x_n^{d_n}}\left(\mathrm{proj}_\lambda \boldsymbol{x}+\hat{x}_{\lambda,j_\lambda}\boldsymbol{e}_{n,\lambda}\right)=\prod_{\kappa=1}^{n} a_{i_\kappa}^{d_\kappa}\frac{\partial^d h}{\partial x_1^{d_1}\dots\partial x_n^{d_n}}\left(\mathrm{proj}_\lambda \boldsymbol{T}(\boldsymbol{x})+x_{\lambda,i_\lambda}\boldsymbol{e}_{n,\lambda}\right)$$
$$-s_{\boldsymbol{i}}\frac{\partial^d h}{\partial x_1^{d_1}\dots\partial x_n^{d_n}}\left(\mathrm{proj}_\lambda \boldsymbol{x}+\hat{x}_{\lambda,i_\lambda}\boldsymbol{e}_{n,\lambda}\right), \tag{6.4}$$

for all $(d_1,d_2,\dots,d_n)\in\mathbb{N}^n$, such that $0\leqslant\sum_{\kappa=1}^n d_\kappa=d\leqslant p$, $d_\lambda\geqslant 1$, if $d>0$, $\boldsymbol{x}\in\partial J_{\mathscr{J}(\boldsymbol{i})}$, $\boldsymbol{i}=(i_1,i_2,\dots,i_n)\in\mathbb{A}_1$, $\lambda=1,2,\dots,n$.

Corollary 6.1 *Let $h\in C^p([0,1]^n)$, $p\geqslant 0$, be a function that interpolates the points of Δ such that all its derivatives of order p satisfy the Lipschitz condition. Consider the case that*

$$Q_{\boldsymbol{i}}(\boldsymbol{x})=H(\boldsymbol{T}_{\boldsymbol{i}}(\boldsymbol{x}))-s_{\boldsymbol{i}}\cdot B(\boldsymbol{x}),\quad \textit{for all } \boldsymbol{x}\in J_{\mathscr{J}(\boldsymbol{i})},\ \boldsymbol{i}\in\mathbb{A}_1,$$

where $H,B\in C^p([0,1]^n)$, such that

$$\frac{\partial^d H}{\partial x_1^{d_1}\dots\partial x_n^{d_n}}(\boldsymbol{x})=\frac{\partial^d h}{\partial x_1^{d_1}\dots\partial x_n^{d_n}}(\boldsymbol{x}),\quad \textit{for } \boldsymbol{x}\in\partial I_{\boldsymbol{i}},$$
$$\frac{\partial^d B}{\partial x_1^{d_1}\dots\partial x_n^{d_n}}(\boldsymbol{x})=\frac{\partial^d h}{\partial x_1^{d_1}\dots\partial x_n^{d_n}}(\boldsymbol{x}),\quad \textit{for } \boldsymbol{x}\in\partial J_{\mathscr{J}(\boldsymbol{i})},$$

for all $(d_1,d_2,\dots,d_n)\in\mathbb{N}^n$, such that $0\leqslant\sum_{\kappa=1}^n d_\kappa=d\leqslant p$, $d_\lambda\geqslant 1$, if $d>0$. The unique attractor G of the corresponding RIFS $\{[0,1]^n\times\mathbb{R},W_{\boldsymbol{i}},P\,;\boldsymbol{i}\in\mathbb{A}_1\}$ is the graph of a function f of class C^p that interpolates the points of Δ.

6.1 An example of C^1 FIFs

Below we give a simple example of a RIFS satisfying the relations of 6.1. We note that in the case where $n=1$, the construction is identical to the cubic Hermite FIFs (see [19]). Let $h\in C^2([0,1]^n)$; we consider the special case where $Q_{\boldsymbol{i}}$ are of the following form:

$$Q_{\boldsymbol{i}}(\boldsymbol{x})=\sum_{k=1}^{n}\sum_{l=0}^{3} r_{k,\boldsymbol{i},l}(\mathrm{proj}_{-k}\boldsymbol{x})x_k^l, \tag{6.5}$$

for all $\boldsymbol{x}=(x_1,x_2,\dots,x_n)\in J_{\mathscr{J}(i)}$, where $r_{k,\boldsymbol{i},l}:\mathrm{proj}_{-k}\left(J_{\mathscr{J}(i)}\right)\to I_{\boldsymbol{i}}$ are functions of class C^1 such that

$$r_{k,\boldsymbol{i},l}(\mathrm{proj}_{-k,\lambda}\boldsymbol{x}+\hat{x}_{\lambda,j_\lambda}\boldsymbol{e}_{n-1,\lambda})=r_{k,\boldsymbol{i},l}\left(\mathrm{proj}_{-k,\lambda}\boldsymbol{x}+\hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n-1,\lambda}\right)=0,$$
$$\frac{\partial r_{k,\boldsymbol{i},l}}{\partial x_\lambda}(\mathrm{proj}_{-k,\lambda}\boldsymbol{x}+\hat{x}_{\lambda,j_\lambda}\boldsymbol{e}_{n-1,\lambda})=\frac{\partial r_{k,\boldsymbol{i},l}}{\partial x_\lambda}\left(\mathrm{proj}_{-k,\lambda}\boldsymbol{x}+\hat{x}_{\lambda,j_\lambda-1}\boldsymbol{e}_{n-1,\lambda}\right)=0,$$

for all $\boldsymbol{x}\in\partial J_{\mathscr{J}(\boldsymbol{i})}$, $\boldsymbol{i}=(i_1,i_2,\dots,i_n)\in\mathbb{A}_1$, $l=0,1,2,3$, $\lambda=1,2,\dots,n-1$, $k=\lambda+1,\lambda+2,\dots,n$. We may easily show that in this case the functional system (6.3)–(6.4) has a unique solution. In particular, for $\lambda=1$, equations (6.3)–(6.4) become

$$\sum_{l=0}^{3} r_{1,\boldsymbol{i},l}\left(\mathrm{proj}_{-1}\boldsymbol{x}+\hat{x}_{1,j_1-1}\boldsymbol{e}_{n,1}\right)\hat{x}_{1,j_1-1}^l=h\left(\mathrm{proj}_1\boldsymbol{T}(\boldsymbol{x})+x_{1,i_1-1}\boldsymbol{e}_{n,1}\right)$$
$$-s_{\boldsymbol{i}}h\left(\mathrm{proj}_1\boldsymbol{x}+\hat{x}_{1,i_1-1}\boldsymbol{e}_{n,1}\right),$$

$$\sum_{l=0}^{3} r_{1,i,l}\left(\operatorname{proj}_{-1}\boldsymbol{x}+\hat{x}_{1,j_1}\boldsymbol{e}_{n,1}\right)\hat{x}^{l}_{1,j_1-1}=h\left(\operatorname{proj}_1\boldsymbol{T}(\boldsymbol{x})+x_{1,i_1}\boldsymbol{e}_{n,1}\right)$$
$$-s_i h\left(\operatorname{proj}_1\boldsymbol{x}+\hat{x}_{1,i_1}\boldsymbol{e}_{n,1}\right),$$
$$\sum_{l=1}^{3} r_{1,i,l}\left(\operatorname{proj}_{-1}\boldsymbol{x}+\hat{x}_{1,j_1}\boldsymbol{e}_{n,1}\right) l\hat{x}^{l-1}_{1,j_1}=a_{i_1}\frac{\partial h}{\partial x_1}\left(\operatorname{proj}_1\boldsymbol{T}(\boldsymbol{x})+x_{1,i_1-1}\boldsymbol{e}_{n,1}\right)$$
$$-s_i\frac{\partial h}{\partial x_1}\left(\operatorname{proj}_1\boldsymbol{x}+\hat{x}_{1,i_1-1}\boldsymbol{e}_{n,1}\right),$$
$$\sum_{l=1}^{3} r_{1,i,l}\left(\operatorname{proj}_{-1}\boldsymbol{x}+\hat{x}_{1,j_1-1}\boldsymbol{e}_{n,1}\right) l\hat{x}^{l-1}_{1,j_1}=a_{i_1}\frac{\partial h}{\partial x_1}\left(\operatorname{proj}_1\boldsymbol{T}(\boldsymbol{x})+x_{1,i_1}\boldsymbol{e}_{n,1}\right)$$
$$-s_i\frac{\partial h}{\partial x_1}\left(\operatorname{proj}_1\boldsymbol{x}+\hat{x}_{1,i_1}\boldsymbol{e}_{n,1}\right).$$

The above system is linear and can always be solved to compute the values $r_{1,i,l}(\operatorname{proj}_{-1}\boldsymbol{x})$, for l = 0, 1, 2, 3, $\boldsymbol{x}\in J_{\mathscr{J}(i)}$, in terms of the values of the function h and its partial derivatives on I_i and $J_{\mathscr{J}(i)}$. In general, for any $\lambda>1$, equations (6.3)–(6.4) give a linear system that can always be solved for $r_{\lambda,i,l}(\operatorname{proj}_{-\lambda}\boldsymbol{x})$, $\boldsymbol{x}\in J_{\mathscr{J}(i)}$, $l=0,1,2,3$, in terms of the values of the function h and its partial derivatives on I_i and $J_{\mathscr{J}(i)}$ and the values $r_{k,i,l}(\operatorname{proj}_{-\lambda}\boldsymbol{x})$, $l=0,1,2,3$, $k=1,2,\ldots,\lambda-1$, which have been computed in the previous steps. For example, for $n=2$, we obtain the following linear systems:

$$\begin{bmatrix}1 & \hat{x}_{j_1-1} & \hat{x}^2_{j_1-1} & \hat{x}^3_{j_1-1}\\ 1 & \hat{x}_{j_1} & \hat{x}^2_{j_1} & \hat{x}^3_{j_1}\\ 0 & 1 & 2\hat{x}_{j_1-1} & 3\hat{x}^2_{j1-1}\\ 0 & 1 & 2\hat{x}_{j_1} & 3\hat{x}^2_{j1}\end{bmatrix}\cdot\begin{bmatrix}r_{1,i,0}(y)\\ r_{1,i,1}(y)\\ r_{1,i,2}(y)\\ r_{1,i,3}(y)\end{bmatrix}=\begin{bmatrix}h(x_{i_1-1},T_{i,2}(y))-s_i h(\hat{x}_{j_1-1},y)\\ h(x_{i_1},T_{i,2}(y))-s_i h(\hat{x}_{j_1},y)\\ a_{i,1}h_x(x_{i_1-1},T_2(y))-s_i h_x(\hat{x}_{j_1-1},y)\\ a_{i,1}h_x(x_{i_1},T_2(y))-s_i h_x(\hat{x}_{j_1},y)\end{bmatrix} \tag{6.6}$$

$$\begin{bmatrix}1 & \hat{y}_{j_2-1} & \hat{y}^2_{j_2-1} & \hat{y}^3_{j_2-1}\\ 1 & \hat{y}_{j_2} & \hat{y}^2_{j_2} & \hat{y}^3_{j_2}\\ 0 & 1 & 2\hat{y}_{j_2-1} & 3\hat{y}^2_{j_2-1}\\ 0 & 1 & 2\hat{y}_{j_2} & 3\hat{y}^2_{j_2}\end{bmatrix}\cdot\begin{bmatrix}r_{2,i,0}(x)\\ r_{2,i,1}(x)\\ r_{2,i,2}(x)\\ r_{2,i,3}(x)\end{bmatrix}=B \tag{6.7}$$

where

$$B=\begin{bmatrix}h(T_{i,1},y_{i_2-1})-s_i h(x,\hat{y}_{j_2-1})-r_{1,i,0}(\hat{y}_{j_2-1})-r_{1,i,1}(\hat{y}_{j_2-1})x-r_{1,i,2}(\hat{y}_{j_2-1})x^2-r_{1,i,3}(\hat{y}_{j_2-1})x^3\\ h(T_{i,1},y_{i_2})-s_i h(x,\hat{y}_{j_2})-r_{1,i,0}(\hat{y}_{j_2})-r_{1,i,1}(\hat{y}_{j_2})x-r_{1,i,2}(\hat{y}_{j_2})x^2-r_{1,i,3}(\hat{y}_{j_2})x^3\\ a_{i,2}h_y(T_{i,1},y_{i_2-1})-s_i h_y(x,\hat{y}_{j_2-1})-r'_{1,i,0}(\hat{y}_{j_2-1})-r'_{1,i,1}(\hat{y}_{j_2-1})x-r'_{1,i,2}(\hat{y}_{j_2-1})x^2-r'_{1,i,3}(\hat{y}_{j_2-1})x^3\\ a_{i,2}h_y(T_{i,1},y_{i_2})-s_i h_y(x,\hat{y}_{j_2})-r'_{1,i,0}(\hat{y}_{j_2})-r'_{1,i,1}(\hat{y}_{j_2})x-r'_{1,i,2}(\hat{y}_{j_2})x^2-r'_{1,i,3}(\hat{y}_{j_2})x^3\end{bmatrix}.$$

In Figure 7, we give one C^1 surface using the above construction.

7 Conclusions

In the present work, some new methods are introduced for the construction of FIFs on arbitrary interpolation points placed on rectangular grids of $\mathbb{R}^n$. We show that these methods generalise the construction given by Barnsley for $n=1$. In spite of all previous

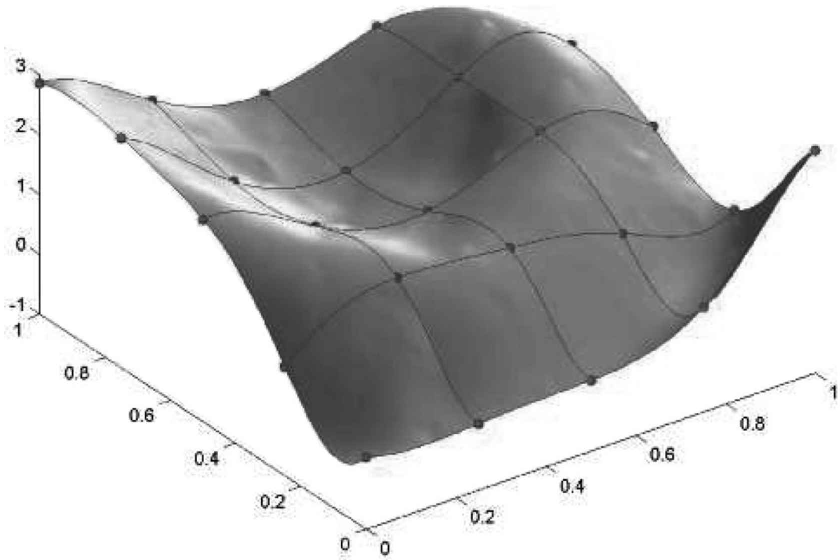

FIGURE 7. A C^1 FIS according to the construction presented in Section 6.1. The black lines are the values of the C^2 interpolation function h.

attempts to generalise Barsnley's construction, which used only the interpolation points to construct the fractal function, our method uses values defined on the boundary of the grid. Therefore, a multivariate interpolant is needed. We presented several interesting examples and believe that many more may be found.

Applications of FIFs to the real line $\mathbb{R}$ may be found in various areas such as one-dimensional signal modelling and computer graphics. However, the lack of a solid mathematical background has prevented scientists and engineers from using FIFs to address many $\mathbb{R}^n$-related problems in the case where $n > 1$, such as the modelling of two-dimensional signals (e.g. images) and generation of rough surfaces for use in computer graphics. We hope that this work may prove helpful for these cases. Finally, we should note that some interesting questions are raised. Recent works have related FIFs with splines and Hermite interpolation polynomials of $\mathbb{R}$. Is there an analogous relation in $\mathbb{R}^n$? How may FIFs be used to generalise splines in any dimension? May we construct orthogonal wavelets using fractal functions (as in [10]) in $\mathbb{R}^n$?

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